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Symmetries and Patterns In Non-Euclidean Settings

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Abstract

From the Megalithic Temples of Malta constructed over 5,500 years ago, to the Pyramid of Djoser in Egypt built some 4,700 years ago, to more recent works of architectural wonder such an the Taj Mahal, the testimonials to the innate human genius for creating beauty through symmetry, color, and patterns abound. Evidently, the mathematical underpinnings of many architectural marvels are mostly rooted in the Euclidean Geometry. Now, as marvelous as these monuments are, one may wonder what would be the concepts of beauty and symmetry in a non-Euclidean universe. It turns out that this is not a far-fetched thought. It is now an established fact (implied, for example, by Einstein's theory of relativity) that the space-time we live in is, in fact, non-Euclidean (or at least more non-Euclidean than Euclidean). Over the last two hundred years, we have come to realize that the notion of distance in the universe is far more complex than what we had always believed. The goal of this thesis is to study the the concepts of symmetry and pattern in a purely topological setting that is independent of any notion of distance.

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Notation

$\mathbb{R}$: The set of real numbers

$\mathbb{R}^n = \{(x_1, x_2, x_3, ..., x_n) : x_i \in \mathbb{R}, 1 \leq i \leq n\}$

$||x|| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$: the Euclidean norm on $\mathbb{R}^n$

$\mathbb{E}^n$: $\mathbb{R}^n$ equipped with the Euclidean norm

$\mathbb{S}^{n-1} = \{x \in \mathbb{E}^n, ||x|| = 1\}$, the $(n-1)$-dimensional geometric sphere

$\mathbb{Z}^+$: the set of positive integers

$|X|$: the cardinality of a set X

$S(X)$: the group of permutations on X

CHAPTER 1

The Discovery of Non-Euclidean Geometries

All changed, changed utterly:
A terrible beauty is born

–*Y.B. Yeats* (*Easter* 1916)

1.1 End of Furor Over the Parallel Postulate

In everyday vernacular, the words *symmetry* and *pattern* are used with no strict definitions. For most of us, the meaning of these words is greatly influenced by the Euclidean spatial structure ingrained in our brains. For example, even as we see a spherical object around us, our brains at once integrate it into the Euclidean world we live in. We forget that a sphere carries its own geometry, where triangles are not ordinary but quite different, and the sum of their three angles can be as large as 540°. Such a belief expressed openly only a couple of centuries ago would have been deemed a heresy. However, as the result of new findings over the last couple of centuries, we are now open to entertain the idea that our universe may actually have as many as 11 dimensions—still a mind-boggling thought! This change in our perception began to take place some 200 years ago.

In 1824, the great master Carl Friedrich Gauss (1777-1855) wrote in a private note to a friend:

> *The assumption that (in a triangle) the sum of the three angles is less than* 180° *leads to a curious geometry, quite different from ours, but thoroughly consistent, which I have developed to my entire satisfaction.*

What Gauss had demonstrated was that there exists a geometry satisfying Euclid's first four axioms (see Appendix) of geometry, but negating the fifth axiom. Gauss was not alone to come up with such a discovery. Independently, and almost contemporaneously with Gauss, N. Lobachevsky (1792-1856), J. Bolyai (1802-1860), and B. Riemann (1826-1866) also discovered non-Euclidean geometries. This discovery stirred the mathematical community, which had religiously believed that the universe was undoubtedly Euclidean and that there was no other possibility. A two-thousand-year-old debate as to whether Euclid's fifth postulate of plane Geometry (*the parallel postulate*) was independent of the first four axioms or not was coming to a close. This started a new era in mathematics where all-truths (=theorems) were to be derived in an axiomatically defined environment, and where a theorem may be true in one environment but false in another.

Riemann forcefully argued that the notion of distance was even more fundamental than Euclid's primitive notions, and that it had to be specified independently. This statement inspired an axiomatization of the notion of distance and subsequently led to the discovery of the concept of topology, where the notion of continuity can stand alone without any reference to distance. In response to the question as to which Geometry is true, Henry Poincaré (perhaps the greatest mathematician who ever lived) crystallized the argument thus:

> *One geometry cannot be more true than the other, it can only be more convenient.*

These monumental discoveries greatly influenced Einstein, who remarked that his theory of relativity would not have been possible without the work of Riemann on what we now call the Riemannian geometry.

1.2 Topology and Topological Spaces

Definition 1.2.1 A *topology* on a set X is a collection τ of subsets of X, called the open sets, satisfying the following axioms:

(1) The union of any collection of open sets is an open set.

(2) The intersection of a finite number of open sets is an open set.

(3) Both X and the empty set $\varnothing$ are open.

If τ is a topology on X, the pair (X, τ) is called a *topological space.* When no confusion is possible, we omit τ and call X a topological space or simply, a space.

Definition 1.2.2 A function f from a space X to a space Y is said to be *continuous* if for each open set G in Y, $f^{-1}(G)$ is open in X. If f is bijective and if both f and f^{-1} are continuous, then f is a *homeomorphism* between X and Y, and X and Y are *homeomorphic* to each other. A homeomorphism of a space X onto itself is an *autohomeomorphism* of X. The autohomeomorphisms of a space X form a group (under composition) which we denote by $\mathcal{A}(X)$.

Definition 1.2.3 If X is a non-empty set, a function $d : X \times X \to \mathbb{R}$ is called a *metric* on X if for all $x, y, z \in X$, the following conditions are satisfied:

(1) d is positive definite: $d(x, x) = 0$, $d(x, y) > 0$ if $x \neq y$

(2) d is symmetric: $d(x, y) = d(y, x)$

(3) d satisfies the triangle inequality: $d(x, y) + d(y, z) \geq d(x, z)$.

If d is a metric on X, the pair (X, d) (or simply X when d is understood from the context) is called a *metric space.* For any $a \in X$ and any $\epsilon > 0$, the set $S(a, \epsilon) = \{x \in X : d(x, a) < \epsilon\}$ is called a *sphere* centered at a of radius ϵ.

A metric on a set X *generates* a topology on X as follows: Call a subset G of X open if G can be expressed as a union of a collection of spheres. A topology on X is said to be *metrizable* if it can be generated by a metric on X.

If (X, d) is a metric space and Y a subset of X, the restriction of d to $Y \times Y$ is a metric on Y; Y equipped with this metric is called a *subspace* of X.

Caution 1.2.4 The term sphere and notation $S(a, \epsilon)$ defined here are not to be confused with the n-dimensional geometric sphere $\mathbb{S}^n$. We also note that in the everyday vernacular, the word sphere is used exclusively for the surface $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$.

Definition 1.2.5 A topology on a set X is called metrizable if it can be generated by a metric on X. Two metrics d_1 and d_2 on X are *topologically equivalent* if they both generate the same topology on X.

A metrizable topology on a set (with two or more elements) can be generated by infinitely many distinct metrics. Also, the number of distinct topologies on a set X increases as $|X|$ gets larger.

1.3. Symmetries and Isometries

Definition 1.3.1 A bijection T of a metric space (X, d) onto itself is called an *isometry* of X if $d(T(p), T(q)) = d(p, q)$ for all $p, q \in X$. A subspace Y of X is said to be *invariant* under T if $T(Y) = Y$. The *symmetry group* G of Y *relative* to X is the set of all isometries of X for which Y is invariant. The symmetry group of Y (relative to Y) is simply the *group of isometries* on Y.

Evidently, the symmetry group of Y relative to X is a subgroup of the symmetry group of Y. The following example shows that the two groups may be different from each other.

Example 1.3.2 For both $D = \{(x,y) \in \mathbb{E}^2 : x^2 + y^2 = 1\}$ and $S = \{(x,y) \in \mathbb{E}^2 : |x| \leq 1 \text{ and } |y| \leq 1\}$, the symmetry groups of D in S and of S in $\mathbb{E}^2$ are both isomorphic to the dihedral group of order 8. On the other hand, the symmetry group of D in $\mathbb{E}^2$ is isomorphic to the infinite dihedral group of the symmetries of $\mathbb{S}^1$.

1.4 Colorings

Definition 1.4.1 For nonempty sets X and Y, a function $f : X \to Y$ is called a *coloring* of X (a Y-coloring of X, for clarity) and the elements of Y are called *colors.* For any $y \in Y$, the set $f^{-1}(y)$ is called the *y-patch* (briefly, a patch) in X. The nonempty patches in X constitute a partition of X. Conversely, any partition of X induces a coloring of X where the patches are precisely the cells constituting the partition.

1.5 Patterns

Definition 1.5.1 Let X be a set and G be a fixed subgroup of the permutation group $S(X)$. Two colorings f_1 and f_2 of X with colors in Y are said to be *G-equivalent* if for some α in G, $f_1 \circ \alpha = f_2$.

Since G is a group, the relation of G-equivalence on the set of all Y-colorings of X is an equivalence relation.

Definition 1.5.2 An equivalence class of G-equivalent colorings of X is called a *G-pattern* (or simply a pattern when some specific G is implied), and the equivalence class containing a particular coloring f is denoted by $[f, G]$. Two G-equivalent colorings are said to *display* the same G-pattern.

1.6 Group Actions

If G is a subgroup of $S(X)$, the map

$$G \times X \to X, \ (g, x) \to g(x)$$

is called an *action* of G on X, and we say that G *acts* on X. The following properties of a group action are trivially true.

(1) $e(x) = x$ for all $x \in X$, where e denotes the identity element of G

(2) $(g \cdot h)(x) = g(h(x))$ for all $g, h \in G$.

Definition 1.6.1 G is said to *act transitively* on X if for all $x, y \in X$, $g(x) = y$ for some $g \in G$. If k is a positive integer with $k \leq |X|$, then the action of G on X is called *k-transitive* if, for every pair of k-tuples $(x_1, ..., x_k)$ and $(y_1, ..., y_k)$, with each of the two k-tuples having distinct entries, there is $g \in G$ such that, for each $i = 1, 2, \ldots, k$, $g(x_i) = y_i$.

Given a positive integer $k \leq |X|$, the problem of determining the subgroups G of $S(X)$ for which G-action is k-transitive is of great importance in Modern Algebra.

All theorems proved in chapters 2 and 3 are entirely new.

CHAPTER 2

Symmetries Under Non-Euclidean Metrics

2.1 The Concept of a Midpoint

Recall that in the linear space $\mathbb{R}^n$, the midpoint of a given pair of points x and y is defined as $\frac{x+y}{2}$. This makes sense because $\frac{x+y}{2}$ is midway between x and y on the line segment joining them. But what if we consider a non-Euclidean metric on $\mathbb{R}^n$? More generally, what if we were to disregard the linear structure as well as the Euclidean norm of $\mathbb{R}^n$, and instead, impose on it a different but equivalent metric? The new metric would retain the topological structure of $\mathbb{R}^n$, but gone would be the notions of a line segment and of adding x and y. In this new setting, what would replace the notion of a 'midpoint'? Here is a plausible answer:

Definition 2.1.1 Let (X, d) be a metric space. For any $x, y, z \in X$, call z a *midpoint* of x and y if $d(x, z) = d(z, y) = \frac{1}{2}d(x, y)$.

The imposed equalities in this definition capture the essence of the term midpoint as used in the Euclidean setting. However, it is easy to show by examples that this new definition does not prevent a pair of distinct points from having infinitely many distinct midpoints, while another pair may have no midpoints at all. For a pair of antipodal points on $\mathbb{S}^n$, the set of midpoints is $\mathbb{S}^{n-1}$. Next, we consider the concept of midpoint on $\mathbb{R}$ under a non-Euclidean metric.

2.2 Symmetry Groups of Non-Euclidean Metrics on $\mathbb{R}$

Consider the metric d_3 on $\mathbb{R}$, defined by $d_3(x, y) = |x^3 - y^3|$. It is easy to prove that d_3 is indeed a metric on $\mathbb{R}$, and that it is topologically equivalent to the Euclidean metric. Furthermore, d_3 retains another property of the Euclidean metric, namely, if $x < y < z$, then $d_3(x, y) + d_3(y, z) = d_3(x, z)$. However, the point 1 is no longer the midpoint of 0 and 2 because $d_3(0, 1) = 1 < d_3(1, 2) = 7$. Applying the definition of a midpoint we conceived for metric spaces, we can check that the midpoint of a and b is $\left(\frac{a^3+b^3}{2}\right)^{\frac{1}{3}}$, which does not conform to the Euclidean view of midpoint. We can easily check that the symmetry group H of the metric space $(\mathbb{R}, d_3)$ contains no non-trivial translations and that this group is only of order 2. In contrast, the isometry group of $\mathbb{R}$ under the Euclidean metric is the uncountable dihedral group generated by the translations $T_a : x \to x + a$, $a \in \mathbb{R}$, and the reflection $\lambda : x \to -x$.

2.3 Symmetry Groups of Non-Euclidean Metrics on $\mathbb{R}^n$

The definition of the metric d_3 on $\mathbb{R}$ can be easily generalized to obtain metrics on $\mathbb{R}^n$ that are topologically equivalent to the Euclidean metric, but have much smaller symmetry groups than those under the the Euclidean metric.

CHAPTER 3

Patterns Under Group Actions

3.1 Properties of Group Actions

The goal of this chapter is to study the G-patterns of colorings for some familiar subgroups G of $S(X)$.

If X is Y-colored and if $|Y| = 1$, then there is only one coloring on X, and the notion of patterns of Y-colorings of X becomes trivial. Hence we assume that $|Y| > 1$, and indeed we do not assume that $|Y|$ is finite.

Important properties of G-patterns are recorded in the following three theorems.

Theorem 3.1.1 For two colorings f_1 and f_2 to display the same pattern, it is necessary (but not sufficient) that for each color y in Y, the y-patches $f_1^{-1}(y)$ and $f_2^{-1}(y)$ have the same cardinality.

Theorem 3.1.2 If G_1 and G_2 are subgroups of $S(X)$ and if $G_1 \subseteq G_2$, then for each coloring f of X, $[f, G_1] \subseteq [f, G_2]$.

Observe that if $G = \{e\}$, then $[f, G] = \{f\}$ for any coloring f. Thus, in this case, no two distinct G-colorings display the same pattern. The following theorem deals with the other extreme when $G = S(X)$.

Theorem 3.1.3 Two colorings f_1 and f_2 are $S(X)$-equivalent if and only if $|f_1^{-1}(y)| = |f_2^{-1}(y)|$ for all $y \in Y$.

Proof The 'only if' part follows trivially from the definition of G-equivalence. To prove the 'if' part, construct a bijection α on X for which $f_2 = f_1 \circ \alpha$ as follows: Since $|f_1^{-1}(y)| = |f_2^{-1}(y)|$, there exists a bijection α_y from $f_1^{-1}(y)$ onto $f_2^{-1}(y)$ for each y. Now, let α be the function on X whose restriction to each $f_1^{-1}(y)$ is α_y. Since the patches f_1^{-1} are pairwise disjoint, α is well-defined and is a bijection on X. The construction of α ensures that $f_1 = f_2 \circ \alpha$. □

Obviously, $S(X)$ is k-transitive for any positive integer k with $k \leq |X|$, and the notion of $S(X)$-equivalence of two colorings f and g reduces to a simple test of whether each f-patch of a certain color has the same cardinality as the g-patch of the same color. It is easy to construct examples of colorings that are $S(X)$-equivalent but not $A(X)$-equivalent [Here $A(X)$ is the alternating subgroup of $S(X)$].

3.2 Patterns of Continuous Maps

Let X and Y be arbitrary topological spaces and let $\mathcal{C}(X, Y)$ denote the set of all continuous maps (henceforth maps) from X to Y. We call each element of $\mathcal{C}(X, Y)$ a *continuous coloring* of X with colors Y. As stated earlier, $\mathcal{A}(X)$ denotes the group of autohomeomorphisms of X. We list some important properties of the continuous colorings of X.

(1) If for each y in Y, the set $\{y\}$ is a closed set in Y (i.e., if Y is a T_1 space, in topological lingo), then for each map f, the f-patches are closed sets in X. Additionally, if X is a compact Hausdorff space, then all the patches are compact sets.

(2) Any single element of X may receive several different colors.

(3) Call a subset F of X a *lump* if F is the union of a collection of patches. If X is normal, Hausdorff (but not necessarily compact) and if Y is the Euclidean line $\mathbb{E}^1$, then any two disjoint closed lumps in X can be separated by disjoint open sets (Urysohn's Lemma). Thus, the family of all patches is not entirely chaotic.

3.3 Patterns of Maps on $\mathbb{R}$

Throughout this section, $\mathbb{R}$ denotes the real line with its topology generated by the open intervals; the word 'map' means a continuous function from $\mathbb{R}$ into $\mathbb{R}$, and G denotes the autohomeomorphism group of $\mathbb{R}$.

The group G has been studied thoroughly in literature. The following properties of G can be easily proved.

(P1) The elements of G are precisely the strictly monotone surjections on $\mathbb{R}$. An element α of G is called a *flow* if α is strictly increasing, and a *reversal* if α is strictly decreasing.

(P2) The set F of flows is a normal subgroup of index 2 in G, and the reversals form the other coset of F.

(P3) F is 2-transitive but not 3-transitive on $\mathbb{R}$.

Theorem 3.3.1 For any $\alpha \in G$, the G-pattern of α is the entire group G.

Proof If $\beta \in G$, then $\beta = \alpha(\alpha^{-1}(\beta))$, whence the G-pattern of α contains G. Conversely, if $g = \alpha \circ \beta$ for some β in G, then g is a composition of two autohomeomorphisms, whence g is necessarily an autohomeomorphism. □

The goal of this section is to study the patterns of F-equivalent maps. Evidently, if f and g are G-equivalent, then either they are F-equivalent or f and $-g$ are

F-equivalent. Thus, the study of G-equivalence reduces to that of F-equivalence. Whenever f and g are F-equivalent, we write $f \sim g$. The notion of F-equivalence of maps can be expressed in a diagram: Two maps f and g are F-equivalent if and only if there exists α in F for which the following diagram commutes.

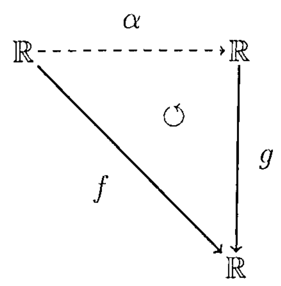

Figure 3.3.2 Commutative diagram for F-equivalence of maps

The following statements for a map f are trivially true. [The word 'interval' here includes open, closed, half-open, bounded, or unbounded intervals.]

(O1) If f is monotone on two overlapping open intervals I_1 and I_2, then it is also monotone on $I_1 \cup I_2$.

(O2) If f is monotone on an interval I, then there is a largest interval I^* containing I such that f is monotone on I^*.

(O3) If f is monotone on an interval I, then it is also monotone of the closure of I (by the continuity condition on f).

The following theorem tells us how the intervals on which f is monotonic correspond to those on which g is monotonic.

Theorem 3.3.3 Suppose $g = f \circ \alpha$ for some α in F. Then, for each interval I on which g is monotonic, f is monotonic on $\alpha(I)$.

Proof Since α is a flow, the hypotheses $g = f \circ \alpha$ and g monotonic on I imply the desired conclusion. □

We give two examples of F-equivalent maps, and an example of two G-equivalent maps under a reversal.

Example 3.3.4 $f(x) = \sin x$, and $g(x) = \cos x$ are G-equivalent. [$g = f \circ T$ where T is the translation $T(x) = x + \frac{\pi}{2}$. In fact, f and g are F-equivalent under a much smaller group than G, namely the cyclic subgroup $\langle T \rangle$ of G generated by T.]

Example 3.3.5 In Example 3.3.4, replace g by $g_1(x) = \sin(x^3)$ and let $T_1(x) = x^3$. Then g_1 and f are F-equivalent [$g_1 = f \circ T_1$].

Example 3.3.6 The functions $f(x) = e^x$ and $g(x) = e^{-x}$ are G-equivalent [$f \circ \alpha = g$, where α is the reversal $x \to -x$].

To further examine the concept of F-equivalence of maps, we introduce a new concept relating to a local property of maps.

3.4 Zig-Zag Maps

Definition Suppose there exists an increasing sequence $a_1 < a_2 < a_3 < \ldots$ of real numbers converging to a point p such that f is strictly monotone on each interval $I_i = [a_i, a_{i+1}]$ with opposite monotonicity on any two of consecutive intervals in this collection. Then f is said to *zig-zag* to p from the left. Replacing the 'increasing sequence' by a 'decreasing sequence' defines the concept of a map *zig-zagging* from the right of p. The meaning of a map f zig-zagging to p from both sides is now obvious.

Example 3.4.1 The following map zig-zags to 0 from both sides:

$$f(x) = \begin{cases} x\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

Example 3.4.2 This is an example of a piecewise linear map zig-zagging to 0 from both sides. For $x > 0$, the graph of f is defined as follows: For $x > 1$, $f(x) = x$. Also, for each positive integer n, the graph of f is made of the two line segments joining the point $(\frac{1}{2n}, -\frac{1}{2n})$ to the points $(\frac{1}{2n-1}, \frac{1}{2n-1})$ and $(\frac{1}{2n+1}, \frac{1}{2n+1})$. Finally, let $f(0) = 0$ and $f(x) = f(|x|)$ for $x < 0$.

Given a point $q \in \mathbb{R}$ and a sequence $\{p_i\}$ converging to q, one can construct a map f that zig-zags to each p_i from both sides.

For a map f, $L(f)$, $R(f)$, and $D(f)$ respectively denote the set of points p to which f zig-zags only from the left, only from the right, and from both sides. If two maps f and g are F-equivalent, how do the three sets $L(f)$, $R(f)$, and $D(f)$ relate to the corresponding sets for g? To answer this question, we recall a concept from the theory of partially ordered sets.

Definition Two partially ordered sets $(P_1, \precsim_1)$ and $(P_2, \precsim_2)$ are said to be order-isomorphic to each other is there exists a bijection h from P_1 onto P_2 such that

$$x \precsim_1 y \text{ iff } h(x) \precsim_2 h(y).$$

Theorem 3.4.3 If two maps f and g are F-equivalent, then there exists an order-isomorphism between the sets $L(f) \cup R(f) \cup D(f)$ and $L(g) \cup R(g) \cup D(g)$ that maps $L(f)$ onto $L(g)$, $R(f)$ onto $R(g)$, and $D(f)$ onto $D(g)$.

Example 3.4.4 Consider a map f whose graph on the interval $[-1, 1]$ defined as follows: For each even positive integer n, the graph of f consists of the line segment joining the point $(\frac{1}{n}, -\frac{1}{n})$ to the points $(\frac{1}{n-1}, \frac{1}{n-1})$ and $(\frac{1}{n+1}, \frac{1}{n+1})$. Let $f(0) = 0$ and let the graph of f on the interval $[-1, 0)$ be the reflection over the y-axis of its graph on $(0, 1]$. Also, let $f(x) = 1$ for all $x \notin (-1, 1)$. Then f zig-zags to 0 from both sides.

Using the construction given in Example 3.4.4, we can create a map g zig-zagging to infinitely many points. For any such map g, the set of points to which g zig-zags must be order-isomorphic to the set B consisting of those points to which $h = f \circ \alpha$ zig-zags (for an autohomeomorphism α). Furthermore, if α is a flow, then this isomorphism must match $L(g)$ with $L(f)$, $R(g)$ with $R(f)$, and $D(g)$ with $D(h)$.

The following example illustrates that for two functions f and g with similar shapes, there may exist a homeomorphism α such that $f = g \circ \alpha$.

Example 3.4.5 Consider the functions $f(x) = |x|$ and $g(x) = x^2$. Let

$$\alpha(x) = \begin{cases} \sqrt{x} & \text{if } x \geq 0 \\ -\sqrt{|x|} & \text{if } x < 0 \end{cases}.$$

The following shows that f is homeomorphic to g:

$$(g \circ \alpha)(x) = \begin{cases} (\sqrt{x})^2 & \text{if } x \geq 0 \\ (-\sqrt{|x|})^2 & \text{if } x < 0 \end{cases} = |x| = f(x).$$

CHAPTER 4

Sphere Packings

4.1 The Sphere Packing Problem

In this chapter, we use previously defined notions of pattern and symmetry to discuss the celebrated Sphere Packing Problem in a non-Euclidean setting. As has been established in previous chapters, non-Euclidean geometries often create different symmetries than the Euclidean Geometry. As a starting point to talk about these interesting symmetries, we first discuss the Sphere Packing Problem in its usual Euclidean form.

Posed by Johannes Kepler in 1611, the original Sphere Packing Problem (SPP) asks for the largest portion of $\mathbb{R}^n$ that can be covered by disjoint congruent n-dimensional spheres. The SPP has been solved in several dimensions. Trivially, in one dimension, a sphere is an open line segment of length $2r$, and thus any arrangement of spheres with touching boundaries maximizes the density. In 1940, Hungarian mathematician Lazlo Fejes Tóth provided the first complete proof that a hexagonal packing, in which spheres cover about 91% of the plane, is the densest packing of $\mathbb{R}^2$. Gauss proved that the face-centered cubic packing is the densest lattice packing of $\mathbb{R}^3$. Additionally, the densest lattice packings are known in $\mathbb{R}^4$ through $\mathbb{R}^8$ and in $\mathbb{R}^{24}$.

As recently as 1998, American mathematician Thomas Hales posted a proof by computer that the face-centered cubic packing is the densest of all packings (both regular and non-regular) of $\mathbb{R}^3$. Faced with skepticism from the mathematical community, Hales set out to produce a formal proof of his findings, which was completed in August 2014.

In the following definitions, we introduce terminology to formally discuss sphere packings.

Definition 4.1.1

$x_1, x_2, x_3 \ldots$: a fixed infinite sequence of points in $\mathbb{E}^n$

r: a fixed positive real number

$S_o = S(0, r)$

$S_i = S_0 + i$

$\mathcal{P} = \{S_i\}$ is called a *lattice packing* of $\mathbb{R}^n$ if the S_i's are pairwise disjoint and the x_i's form a group under addition (which has dimension n).

s: any positive real number

H_s: the right-closed, left-open n-cube of side s centered at 0

The *density* of the packing $\{S_i\}$ is the fraction of $\mathbb{E}^n$ covered by the spheres in the packing. To make this statement precise, consider a packing $\mathcal{P}$.

(1) Add up the volumes of all S_i in the packing $\mathcal{P}$ that intersect H_s and divide this number the the volume of H_s (i.e., s^n). Call this ratio $\rho_+(\mathcal{P}, H_s)$.

(2) Similarly, add up the volumes of all S_i in the packing $\mathcal{P}$ that are contained in H_s, and divide the sum by s^n. Call this ratio $\rho_-(\mathcal{P}, \mathcal{H}_f)$.

The *lower density* of $\mathcal{P}$ is defined as

$$\rho_-(\mathcal{P}) = \lim_{s\to\infty} \inf \rho_-(\mathcal{P}, H_s).$$

Likewise, the *upper density* of $\mathcal{P}$ is

$$\rho_+(\mathcal{P}) = \lim_{s\to\infty} \sup \rho_+(\mathcal{P}, H_s).$$

Obviously, $\rho_-(\mathcal{P}) \leq \rho_+(\mathcal{P})$ and equality holds for a lattice packing. When $\rho_-(\mathcal{P}) = \rho_+(\mathcal{P})$, the common value is called the density of the packing $\mathcal{P}$. In $\mathbb{E}^2$, the density is $\frac{\pi}{2\sqrt{3}}$ for the lattice generated by $(2, 0)$ and $(1, \sqrt{3})$, whereas it is $\frac{\pi}{4}$ for the lattice generated by $(2, 0)$ and $(0, 2)$. In each case, all spheres have radius 1.

To talk about density, we must define the volume of a sphere. The volume of an n-dimensional Euclidean sphere is given by

$$V_n(r) = \frac{\pi^{\frac{n}{2}} r^n}{\Gamma(\frac{n}{2}+1)}$$

which is derived through integration by cross-section slicing. The volumes of $S(a,r)$ in the first eight dimensions are:

$$\begin{aligned}
V_1(r) &= 2r \\
V_2(r) &= \pi r^2 \\
V_3(r) &= \frac{4}{3}\pi r^3 \\
V_4(r) &= \frac{1}{2}\pi^2 r^4 \\
V_5(r) &= \frac{8}{15}\pi^2 r^5 \\
V_6(r) &= \frac{1}{6}\pi^3 r^6 \\
V_7(r) &= \frac{16}{105}\pi^3 r^7 \\
V_8(r) &= \frac{1}{24}\pi^4 r^8
\end{aligned}$$

In the following examples, we generate packings of $\mathbb{R}^2$ and $\mathbb{R}^3$ with Euclidean spheres.

Example 4.1.2 A lattice in $\mathbb{R}^2$ is generated by placing the centers of unit circles at integer combinations of $(2,0)$ and $(1,\sqrt{3})$. Consider the equilateral triangle formed by connecting the centers of circles touching each other (shown below). The density of the lattice is found by computing the proportion of shaded area to the total area of the triangle. A simple calculation shows that this density is $\frac{\pi}{2\sqrt{3}} \approx 0.9069$. This packing is called a *hexagonal packing.*

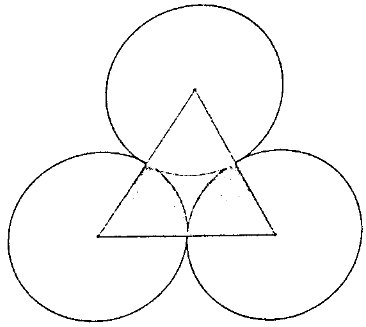

Figure 4.1.3 Hexagonal packing of $\mathbb{R}^2$

Example 4.1.4 Consider the lattice of $\mathbb{R}^3$ that is generated by placing the centers of spheres of radius $\frac{1}{\sqrt{2}}$ at integer combinations of $(1,1,0)$, $(1,0,1)$, and $(0,1,1)$. A similar calculation as in Example 4.1.2 shows that the density is $\frac{\pi}{3\sqrt{2}} \approx 0.7406$. This packing is called *face-centered cubic* (fcc). The face-centered cubic packing generates three repeating layers of spheres (labeled A, B, and C in Figure 4.1.5), where spheres in the third layer are suspended over holes between spheres in the first layer.

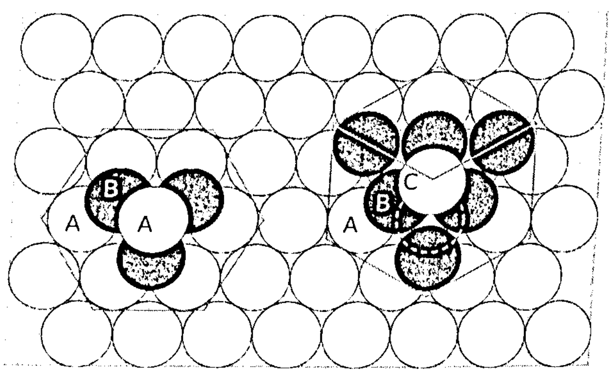

Figure 4.1.5 Hexagonal close packing (left) and face-centered cubic packing (right)

Suppose instead that each sphere in the third layer is suspended directly above a sphere in the first layer. That is, layer C is the same as layer A. This packing is called *hexagonal close* (hcp), and has the same density as fcc.

Roger's upper bound for the density of any packing in $\mathbb{R}^3$ is 0.7796. However, no known packing reaches this density. It was proven by Gauss that fcc is the densest *lattice* packing in $\mathbb{R}^3$. Additionally, about 0.684 is the highest known density for disordered (non-lattice) arrangements in $\mathbb{R}^3$.

4.2 Non-Euclidean Sphere Packings

Now, instead of packing $\mathbb{R}^2$ or $\mathbb{R}^3$ with Euclidean spheres, we pack $\mathbb{R}^2$ and $\mathbb{R}^3$ with spheres determined by the ℓ_p metrics. Let $\|x\|_p = (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{\frac{1}{p}}$, where $p \geq 1$, denote the ℓ_p norm of a point x. Let $d_p(x, y) = \|x - y\|_p$. We briefly write d_p in place of $d_p(x, y)$ when no confusion is possible. Let $S_p(a, r) = \{x \in \mathbb{R}^n : \|x - a\|_p < r\}$, the sphere of radius r centered at a. Briefly, S_p denotes a sphere of radius 1 when the center is arbitrary.

In $\mathbb{R}^2$, the ℓ_p metrics generate spheres as shown below. It should be obvious that S_1 is a diamond, S_∞ is a square, and S_2 is the usual Euclidean circle.

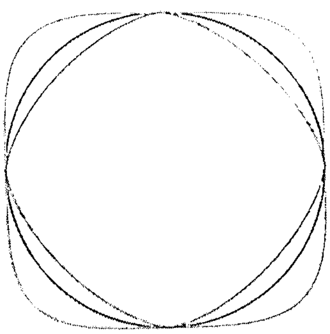

Figure 4.2.1 From the outside in: S_4, S_2, and $S_{\frac{3}{2}}$

In the next example, we generate a packing of $\mathbb{R}^2$ with S_p.

Example 4.2.2 Place the centers of spheres S_p at integer combinations of $(2, 0)$ and $(1, \sqrt[p]{2^p - 1})$. By design, these spheres will not overlap. It will produce a lattice packing as shown below.

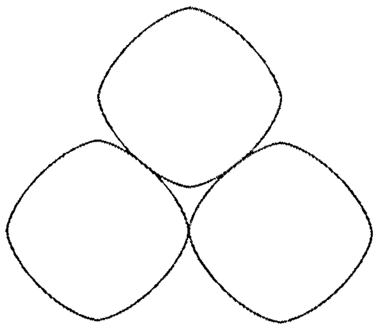

Figure 4.2.3 A lattice packing of $\mathbb{R}^2$ with $S_{\frac{3}{2}}$

The packing described in Example 4.2.2 generates a hierarchy of sphere packings. When $p = 1$ and $p = \infty$, the packing is perfect. When $p = 2$, the hexagonal packing of Euclidean spheres described in Example 4.1.2 is created.

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Appendix

Euclid organized all the mathematical knowledge existing at his time into a total of 13 books called the *Elements.* Elements is recognized as the first comprehensive deductive logical system created. Books I–IV and VI concern plane geometry, books V and VII-X deal with number theory (where numbers are treated geometrically as length, notions such as prime numbers, rational and irrationals are discussed and the infinitude of primes is proved), and books XI–XII are on solid geometry.

Near the beginning, the first book of Elements gives five postulates (axioms) for plane geometry:

"Let the following be postulated":

1. "To draw a straight line from any point to any point."

2. "To produce (extend) a finite straight line continuously in a straight line. "

3. "To describe a circle with any centre and distance [radius]."

4. "That all right angles are equal to one another."

5. The *Parallel Postulate: "That if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on the same side on which are the angles less than two right angles."*