THE 15 PUZZLE AND TOPSPIN

Elizabeth Senac
• 4x4 box with 15 numbers

• Goal is to rearrange the numbers from a random starting arrangement into correct numerical order.

• Can only slide one block at a time.
Definition: The Symmetric Group on n letters is a group with n! elements, where the binary operation is the composition of maps.

\[ S_n \]
Definition: The Alternating Group on n letters is a subgroup of the symmetric group with \( n!/2 \) elements. 

\[ A_n \]
Array notation

$\alpha = \begin{pmatrix} 1 & 2 & \ldots & n \\ \alpha(1) & \alpha(2) & \ldots & \alpha(n) \end{pmatrix}$

MODERN ALGEBRA
MODERN ALGEBRA

Array notation

Cycle notation
• 8 spots

• **Space starts and ends in the 8\text{th} block**

• 7 pieces can move
1 move switches 2 spots

(8 4)

Transposition = cycle of length 2

2 X 4 PUZZLE
If we start with the space in the 8th spot (bottom right hand corner), after how many moves can it return to the 8th spot?

An even number of moves!

2 X 4 PUZZLE
Since each move is 1 transposition, and we are only dealing with even numbers of transpositions, we are in $A_7$ because 7 pieces are able to move. $A_7$ is the group of all even permutations of the symmetric group $S_7$.
We can use the fact that we can only have an even number of transpositions (moves) to tell if a puzzle is solvable before we try to solve it.
Any cycle can be decomposed into transpositions. We have 2 transpositions which is an even number, so this puzzle is solvable.

\[(3 \ 4 \ 7) = (3 \ 4)(4 \ 7)\]

Note: A cycle can be decomposed different ways, but if its decomposition has an even number of transpositions, it will always have an even number of transpositions. Same for odd.
Thm: For $n \geq 3$, consecutive 3-cycles of the form $(i \ i+1 \ i+2)$ where $1 \leq i \leq n-2$, generate $A_n$.

For $A_7$ this means if we can get from the identity to

$(1 \ 2 \ 3)$
$(2 \ 3 \ 4)$
$(3 \ 4 \ 5)$
$(4 \ 5 \ 6)$
$(5 \ 6 \ 7)$

We can get to everything in $A_7$.

ALL OF $A_7$?
Permutations are composed right to left.


(1 2 3)?
Yes, we can get to (2 3 4), (3 4 5), (4 5 6), and (5 6 7) from the identity board. Therefore, we know we can get all of $A_7$. 

WHAT ABOUT THE REST?
Now that we’ve proved our 2 x 4 board is solvable for all even transpositions, how does that help us with our 4 x 4 board?

Think of the 4 x 4 board as 3 overlapping 2 x 4 boards.
A similar argument holds for why we can only have even permutations. It still takes an even number of moves to return the space to the 16th spot.

Now we have to show we can generate all the 3-cycles in $A_{15}$. 

4 X 4
Purple: For (1 2 3), (2 3 4), (3 4 5), (4 5 6), and (5 6 7), just add \((8 12)(12 16)\) to the beginning and \((16 12)(12 8)\) to the end and do the exact same permutations as before.

Red: For (6 7 8), (7 8 9), (8 9 10) and (9 10 11), add \((12 16)\) to the beginning and \((16 12)\) to the end and then treat the permutations the same except replace 2 with 6, 3 with 7, 4 with 8, and so on.

Aqua: For (10 11 12), (11 12 13), (12 13 14), and (13 14 15), treat the permutations the same, but replace 2 with 10, 3 with 11, etc.

**ALL OF \(A_{15}\)?**
SOLVABLE?

\[
\alpha = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
1 & 2 & 4 & 7 & 10 & 6 & 3 & 8 & 5 & 9 & 12 & 15 & 13 & 14 & 11 \\
\end{pmatrix}
\]

\[(3 4 7)(5 10 9)(11 12 15)\]

\[= (3 4)(4 7)(5 10)(10 9)(11 12)(12 15)\]

6 = even # of transpositions (moves), so yes solvable.

\[(2 3 4 7)(5 10 9)(11 12 15)\]

\[= (2 3)(3 4)(4 7)(5 10)(10 9)(11 12)(12 15)\]

7 = odd # of transpositions, so not solvable.

SO LVABLE?

Starting from the identity and then scrambling will always yield a solvable puzzle.
• Consists of 20 pieces numbered 1-20

• 4 pieces in the turnstile at a time

• Unscramble the numbers back into numerical order by shifting the numbers around the track and flipping the turnstile
- $s = \text{shift one position clockwise (does not change order of pieces)}$
- $s^{-1} = \text{shift one position counterclockwise (same)}$
- $f = \text{flip the turnstile (produces the permutation (a d)(b c))}$
Start with a scrambled puzzle
Solving 1-16 is not hard, but I came up with algorithms to solve in numerical order.

- \( x = \) # of spots away from last solved piece
- Place last solved piece one spot left of the turnstile
- For \( x = 1 \): solved
- For \( x = 2 \): use the permutation \( fssf^{-1}f \)
- For \( x = 3 \): use the permutation \( fssf^{-1} \) (let this = \( \delta \))
- For \( x = 4 \): use the permutation \( f \)
- Let \( \rho = fssss \)
- For \( 1 = x \text{ mod } 3 \): use the permutation \((\delta)(\rho^{\frac{x-4}{3}})(f)(s^{-1}x^{-4})\)
- For \( 2 = x \text{ mod } 3 \): use the permutation \((\delta)(fs)(\rho^{\frac{x-5}{3}})(f)(s^{-1}x^{-4})\)
- For \( 0 = x \text{ mod } 3 \): use the permutation \((\delta)(ss)(\rho^{\frac{x-6}{3}})(f)(s^{-1}x^{-4})\)

**SOLVE 1-16**
The last 4 digits (17, 18, 19, and 20) are much more difficult to place in order because rearranging them requires moving previously solved pieces.

I found a permutation $\alpha = sfs^{-1}fsfs^{-1}f$ which moves the piece sitting in the leftmost spot of the turnstile 4 spots clockwise (to the right) without changing the order of any other pieces. (1 5 4 3 2)

If you combine this with another permutation $\beta = s^{-13}$ and use the new permutation $\gamma = \alpha\beta\alpha\beta\alpha\beta\alpha$, then you can move a piece 20 spots clockwise (skipping the original position since it is mobile) without changing the order of any other pieces. Thus, you can move any piece 1 spot to the right.

This is important because now we can generate all of $S_{20}$.

**SOLVE 17-20**
Begin: 18 is between the 16 and the 17

\[\alpha\rightarrow\begin{array}{c}
\text{18 moved 4 spots to the right} \\
\beta\rightarrow\text{All pieces shift 3 to the left}
\end{array}\]

Moves the 18 one spot to the right.

Finally, after a total of \(\alpha\beta\alpha\beta\alpha\beta\alpha\), the 18 is after the 17 and all other pieces are in the same beginning order.
Now that we know we can get any piece to move 1 spot to the right (20 spots) with a turnstile of size 4, will it work for any other sizes?

Any puzzle with turnstile of size 4 and total pieces \( y > 5 \) can be solved by our permutations \( \alpha \) and \( \beta \) if \( 0 = y \mod 4 \) or \( 2 = y \mod 4 \) because we can move a piece an odd number of spots away and eventually will hit every position.

Ex. \( 2 = 22 \mod 4 \)
After 1 permutation \( \gamma \), the desired piece moves 1 spot to the left. This is enough to generate all of \( S_n \).

DOES IT ONLY WORK FOR 20 PIECES?
Any puzzle with turnstile of size 4 and total pieces $y > 5$ cannot be solved by our permutations $\alpha$ and $\beta$ if $1 = y \mod 4$ or $3 = y \mod 4$. I am not sure if this means they can never be solved with a turnstile of size 4, but they cannot be solved by using $\alpha$ and $\beta$.

Ex. $1 = 9 \mod 4$
After 1 permutation $\alpha$, if the puzzle started in the identity position, the 1 moves to the right between the 5 and the 6 while all other pieces remain the same. From here if we were to perform the permutation $\alpha \beta$, the 1 would return between the 9 and 2 while all other pieces remain the same. Thus, we return to the identity and we cannot get a piece to move 1 spot in either direction.

**DOES IT ONLY WORK FOR 20 PIECES?**

Thomas W. Judson Abstract Algebra Theory and Applications 2014

Aaron Archer A Modern Treatment of the 15 Puzzle 1999: Cornell University

A huge thanks to Dr. Hoehn for taking the time and energy to bring us this opportunity and help us think of more areas to explore with our projects

Dr. Gash for his book and his help with a proof (and also for teaching Modern Algebra)

😊

BIBLIOGRAPHY AND THANKS