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Problems for Solid Geometry

Goldia S. Repetto

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PROBLEMS FOR SOLID GEOMETRY

by

Goldia Stoner Repetto

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree Master of Science

COLLEGE OF EDUCATION BUTLER UNIVERSITY INDIANAPOLIS 1939
ACKNOWLEDGMENTS

It is a distinct pleasure to acknowledge the assistance and inspiration of teachers and friends. The writer wishes to acknowledge her obligation to the faculty of the Department of Education for suggestions and helpful criticisms. For constant guidance she is deeply grateful to Dr. Albert Rock and Dr. C. Haven McClure; for checking the manual and helpful suggestions she is indebted to Prof. W. M. Brinson; for typing to her close friend, Mrs. J. L. Koebek; and for English criticism to her Phi Phi sister, Mrs. J. E. Crutchfield; and for patience and encouragement to her husband.
The idea of preparing an exercise manual has grown throughout twelve years of experience in teaching solid geometry to classes, ranging in size from fifteen to thirty-five students. During the depression, large classes necessitated constant revision of methods and materials. Interviews with former students and teachers concerning an improved course revealed a desire for a manual to supplement their textbooks. These interviews uncovered the need for functional problems from practical situations in real life. After contacting fourteen outstanding publishers it was learned that exercise books were available only for plane geometry. The publishers acknowledged, however, that an exercise booklet containing life interest problems would be desirable. Students electing solid geometry, though comparatively few in number, are a very important group. The highly specialized individuals who advance the mathematics and the science of the future, emerge from this group. The manual found in Part B of this study contains many practical problems to be used to supplement solid geometry textbooks. The problems are classified according to geometric topics with exercises in each group pertaining to the various fields. Not every student is expected to work each problem. There is an abundance of practice material to provide for the varying abilities and interests of individual students, as well as the variance in classes. These problems can be used in the classroom, for outside preparation, or for special assignment. Review exercises such as true-false, multiple-choice, best-answer, and completion exercises, are found at the end of each unit. If this manual were to be published, this part could be augmented.

Part A of this study endeavors to relate the history of geometry from primitive man to the present time and to discuss briefly the values of geometry and present day ideas, methods, and techniques for courses in geometry. It is the hope that ideas presented in this study will vitalize the courses of solid geometry in the high schools of the twentieth century.

The annotated bibliography found at the end of this thesis does not include all the books and articles checked, but all from which helpful ideas and quotations were taken.

G.S.R.

Anderson, 1939
<table>
<thead>
<tr>
<th>Chapter</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>I. INTRODUCTION</td>
<td>1-6</td>
</tr>
<tr>
<td>Previous material examined</td>
<td>1-14</td>
</tr>
<tr>
<td>Need revealed for functional exercises</td>
<td>2</td>
</tr>
<tr>
<td>Purpose of the exercise manual</td>
<td>3</td>
</tr>
<tr>
<td>Limitations of this study</td>
<td>4</td>
</tr>
<tr>
<td>Definitions</td>
<td>6</td>
</tr>
<tr>
<td>Summary</td>
<td>6</td>
</tr>
<tr>
<td>II. THE HISTORY OF GEOMETRY</td>
<td>7-33</td>
</tr>
<tr>
<td>Primitive Man</td>
<td>7</td>
</tr>
<tr>
<td>Egyptians and Babylonians</td>
<td>9</td>
</tr>
<tr>
<td>The Greeks</td>
<td>11</td>
</tr>
<tr>
<td>Romans, Hindus, Chinese</td>
<td>23</td>
</tr>
<tr>
<td>Arabs</td>
<td>25</td>
</tr>
<tr>
<td>Modern Times</td>
<td>27</td>
</tr>
<tr>
<td>Summary</td>
<td>33</td>
</tr>
<tr>
<td>III. FUNCTIONAL TRENDS IN PRESENT-DAY GEOMETRY</td>
<td>34-49</td>
</tr>
<tr>
<td>Teaching</td>
<td></td>
</tr>
<tr>
<td>Intuitive and demonstrative geometry</td>
<td>34</td>
</tr>
<tr>
<td>Four values of geometry</td>
<td>35</td>
</tr>
<tr>
<td>Cultural</td>
<td>35</td>
</tr>
<tr>
<td>Preparation</td>
<td>36</td>
</tr>
<tr>
<td>Functional</td>
<td>37</td>
</tr>
<tr>
<td>Disciplinary</td>
<td>38</td>
</tr>
<tr>
<td>Geometry, a way of thinking</td>
<td>41</td>
</tr>
<tr>
<td>Application in non-geometric situations</td>
<td>42</td>
</tr>
<tr>
<td>Summary</td>
<td>48</td>
</tr>
<tr>
<td>IV. PROBLEMS IN TEACHING GEOMETRY</td>
<td>50-57</td>
</tr>
<tr>
<td>Modern Curriculum problems</td>
<td>50</td>
</tr>
<tr>
<td>Vitalizing geometry</td>
<td>54</td>
</tr>
<tr>
<td>Summary</td>
<td>57</td>
</tr>
</tbody>
</table>
TABLE OF CONTENTS (CONTINUED)

<table>
<thead>
<tr>
<th>PART B. PROBLEMS FOR TEACHING SOLID GEOMETRY</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>PREFACE</td>
<td>1</td>
</tr>
<tr>
<td>TABLE OF CONTENTS</td>
<td>11</td>
</tr>
<tr>
<td>Unit</td>
<td></td>
</tr>
<tr>
<td>I. LINES AND PLANES IN SPACE</td>
<td>1-18</td>
</tr>
<tr>
<td>II. POLYHEDRONS, PRISMS, AND CYLINDERS</td>
<td>17-39</td>
</tr>
<tr>
<td>III. PYRAMIDS AND CONES</td>
<td>40-59</td>
</tr>
<tr>
<td>IV. SPHERES</td>
<td>60-75</td>
</tr>
<tr>
<td>ANNOTATED BIBLIOGRAPHY</td>
<td>76-80</td>
</tr>
</tbody>
</table>
PROBLEMS FOR SOLID GEOMETRY

PART A. GEOMETRY

CHAPTER I

INTRODUCTION

Many schools have not made adequate provision for the students who will need solid geometry for advanced studies of professional work. Courses are offered only in the larger high schools because of the small percentage of enrollment electing such a course. Students taking solid geometry are an important group, but relatively few in number. Engineering schools require it for entrance.

Previous Material Examined

All geometry books which publishers presented to the Indiana state textbook adoption committee in 1936 were examined at the request of Arthur Campbell, who was a member of that committee. All of the new books revealed a decided improvement over the older editions. Many of the recent books have sufficiently incomplete proofs of the
theorems to render mere memorizing impossible. Incomplete
proofs are desirable, as they provide adequately for develop-
ment of self-dependence and originality in the student. The
books were rated as to binding, type of print, figures, con-
tent, phrasing, order of topics, presentation, exercises,
and tests. Most new books were divided into units containing
theorems, propositions and exercises pertaining to each of
these particular topics. Many new-type tests were given at
the end of each unit. These exercises were listed as:
- optional constructions,
- more difficult constructions,
- general exercises,
- practical applications,
- numerical exercises,
- true-false statements,
- multiple-choice,
- supplying reasons, and
- exercises.

Careful examination of solid geometry books disclosed
many improvements. The figures are large, shaded drawings
which help to visualize third dimension. The semi-suggestive
method of proofs of theorems motivates many students. Var-
ious devices were used inconspicuously to mark exercises to
provide for the varying abilities of students. The one
thing lacking, however, seemed to be an abundance of prob-
lems which would make the student conscious that geometry
can be and is living. Out of this need grew the inspiration
for the manual of problems in Part 3 of this thesis.
The purpose of this manual grew out of the discussions with classes, former students, and teachers. While new textbooks have improved in many ways, they need to be augmented by other devices. Careful drawings on the blackboard and models bring out the perspective. Stereographs can be used in a small class or as outside reference for a large class to visualize depth or third dimensions. Another device is the exercise manual found in Part B of this study. These exercises can be used to link the textbook material with present day living. Critical thinking and evaluation come from having actual problems to think about and to solve. Such problems may be used to supplement the textbook in the classroom or for daily preparation to care for individual differences and interests of students. This manual can be kept by the teacher and mimeographed copies made of the parts or problems to be used. Any combination of exercises can be made by selecting problems from the desired topics. These exercises can be used effectively in supervised study to give the weak student necessary review of various fundamentals without repetition of the textbook problems. Each student has difficulty with different parts of the work. Rarely does the entire class have the same difficulty, which makes desirable a manual that can be used in many different ways. The teacher will
use these functional problems in accordance with the individual difficulties as they arise. For example, the student may have difficulty in seeing the practical value of knowing how to compute the volume of a sphere, but the goldfish globe in the living room at home or the child's ball or an orange may give him just the incentive that he needs to catch the vision of practical values. Such functional problems extend the view of solid geometry beyond abstract figures. These problems have always existed and can be used effectively in teaching. Their purpose is to provide natural situations which will motivate students. This manual, it is hoped, will vitalize courses in solid geometry for students and teachers.

Limitations of this Study

This manual is limited to solid geometry as it is taught in American high schools today. The four units correspond to those of the adopted textbooks. Each unit is made up of functional problems and review exercises. The problems are classified by such terms in solid geometry as volume or area of specific figures. There are many functional problems under each heading, but the problem may be dealing with engineering construction, with surveying, or with other different fields. This method was used to make it easy for student and teacher to find quickly any kind of desired exercise under geometric headings.
Definitions of Terms Used in this Study

Geometry is used in entire book to mean plane or solid geometry. Historians have used this term in this way because plane and solid geometry use the same kind of logical reasoning. Besides solid geometry, geometric reasoning and plane geometry is a prerequisite for solid geometry, which uses the plane in many of its proofs.

Functional problems are exercises that provide natural situations which motivate students because they provide something vital with human appeal. They are real problems that will challenge the student, since they appear normally in his everyday environment. A problem is functional for the student if it is a problem in a life situation which someone will likely have to solve and in solving will use the same method used by the student. Many problems have been checked according to The Winton Simplified Dictionary's meaning of practical as "capable of being put to use", "having useful ends in view", or "capable of useful action". Most of the exercises are concrete problems meaning "perceptible by the senses; specific, not general, in application". These functional problems (that is, operative in situations or real life experiences) are essential problems to someone of present

living. Every problem is not vital to every person or of interest to all students. The manual found in Part B of this study gives an abundance of exercises which can be used successfully by an enthusiastic, resourceful teacher. It will, moreover, aid the teacher who is a beginner or who may be at a loss to know how to vivify or enrich the average solid geometry textbook offering.

In preparing this manual it was thought that interest would be added if the history of geometry could be related briefly. It was impossible to separate entirely the history of algebra and arithmetic from geometry, since there is an overlapping in their development. Part A of this study relates the history of geometry from the day of primitive man to the present time, and sets forth the values of geometry and present day ideas which will vitalise the courses of solid geometry in the high schools of the twentieth century.

This chapter has commented on the material which created an interest for writing a manual containing many functional problems to be used in teaching solid geometry. Definition of terms explained the meaning of words as they are to be interpreted in this study. An explanation was given for Part A with the hope that the entire study would be more fully appreciated.
CHAPTER II

THE HISTORY OF GEOMETRY

In this study it is impossible to relate the history of geometry without bringing in some of the history of algebra and arithmetic, since there is overlapping in their development.

Mathematics and its practical applications have always been a very accurate gauge of the social conditions and mental development of a people. Primitive man must have begun with mathematical concepts not far in advance of those of animals, but with man, counting was only the beginning and not the end of his abilities. Many lower animals, along with primitive man, seem to share the ability to count or discriminate multiplicity in small groups. Shepherds’ dogs are said to know when one of the flock is missing. Birds seem to discover when an egg is missing from the nest. Animals seem to recognize efficiency in the use of geometric forms; as, for instance, the hexagonal cells of the honey comb, the symmetry of many birds’ nests and the regular bracing used in the spider’s web.
Another interesting observation of mathematical sense in animals is the path of the wild animals in crossing a gully. They use the zig-zag path which civil engineers have used most effectively today for building mountain roads.

Mathematical law and proportion existed in the natural world long before the existence of human life. These relationships were not invented by man, but he has found many ways and means of discovering, comparing, and expressing them. This knowledge is that which we call mathematics or the sense of exact relationships. According to the opinion of Keyser, mathematical begins with the ability to discriminate multiplicity, as in the recognition of two objects instead of one, and is therefore the most primitive achievement of mind.

Levi Leonard Conant says that the number words of primitive tribes are entirely concrete or associated with certain objects which they used in counting. Our numbers, such as two and three, carry a meaning without being connected with any objects. The word used for three by these


primitive people of Africa or Australia might mean three sticks, three fingers, or three pebbles, according to what had best served their purpose in counting.

Egyptian mathematics was well developed before the European countries had emerged from their primitive savagery. Its use in land measurement led to methods of surveying, because the overflow each year of the Nile River obliterated the landmarks. Their worship of the sun god led to astronomical observations, calculations of the seasons, geometric construction of temples, and cutting of building stone. These buildings and monuments indicate an accurate use of mathematics in surveying and leveling. They show a knowledge of the laws of the lever, the inclined plane, and the law of permanent construction. The Egyptian temple builders were aware of the rule still used by surveyors in laying off a perpendicular. Today we take eight links of a surveyor’s chain, place the ends of the chain four links apart, and stretch it with a pin at the fifth link. This forms a right-angled triangle with sides 3, 4, 5. The Egyptian surveyors or “rope stretchers” by means of ropes knotted and stretched to give the same ratio were able to establish levels, find exact east and west line

from the north pole and perform many feats of indirect measurement. Among the Egyptians and Babylonians the construction of figures of religious significance led up to a formal geometry of divination, which recognized triangles, quadrilaterals, right triangles, circles inscribed in the regular hexagon and the division of the circumference into three hundred and sixty degrees. Stereometric problems, such as finding the contents of granaries, are found in Ahmes' Papyrus, but not much is to be learned from his statements, since no account is given relative to the shape of the storehouses. Egyptian and Babylonian surveyors never advanced beyond the intuitive stage in which the measuring of tangible objects was the chief consideration. Whether such knowledge of intuitive geometry was common knowledge among other races is hard to say, for so few early records have been preserved. Doubtless, this is why Egypt has been called the birthplace of geometry.

Intuitive geometry is probably the oldest human conception of mathematics other than the most primitive recognition of number. The word "geometry" is derived from two Greek words meaning "to measure the earth." The subject

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was originally mensuration based upon simple intuition. This kind of mensuration is entirely different from geometry as conceived today. With us the purpose is primarily the logical demonstration of some particular truth concerning a given figure as a triangle, a circle, or a general plane, or a solid figure.

It is to the Greeks that we owe the great development of geometry beyond the intuitive type. The indebtedness of the Greeks to the Egyptians is well summarized by Gow:

It remains only to cite the universal testimony of Greek writers, that Greek geometry was, in the first instance, derived from Egypt, and that the latter country remained for many years afterward the chief source of mathematical teaching. The statement of Herodotus on this subject has already been cited. So also in Plato's 'Phaedrus', Socrates is made to say that the Egyptian god Thoth first invented arithmetic and geometry and astronomy. Aristotle also ('Metaphysics', I, 1) admits that geometry was originally in Egypt; and Pausanias expressly declares that Thales studied there. Much later Diodorus (70 B.C.) reports on Egyptian tradition that geometry and astronomy were the inventions of Egypt, and says that the Egyptian priests claimed Solon, Pythagoras, Plato, Democritus, Sophocles of Chios, and Socrates as their pupils. Strabo gives further details about the visits of Plato and Socrates... Beyond question, Egyptian geometry, such as it was, was eagerly studied by the early Greek philosophers, and was the germ from which, in their hands, grew that magnificent science to which every Englishman is indebted for his first lessons in right seeing and thinking.\footnote{James Gow, A Short History of Mathematics, New York: G. E. Stechert and Co., Reprint 1923, p. 131.}
The Greeks were the first to create a science of geometry. No one thought of learning new relationships by reasoning rather than observations until the time of Thales who lived about 600 B.C. Thales studied in Egypt and acquired the mathematical lore of the priesthood in that country. He finally established a school in Asia Minor, where the first important scientific investigations of geometry were made. The structure of the universe and the astronomical observations are the well-known theorems which Thales seemed to have been chiefly devoted. His chief astronomical teachings are that the year is 365 days, that the moon is illuminated by the sun, and that the earth is spherical.

James Gow, in A Short History of Greek Mathematics attributes to Thales the following five geometrical theorems:

1. The circle is bisected by its diameter.
2. The angles at the base of an isosceles triangle are equal.
3. If two straight lines cut one another the opposite angles are equal.
4. The angle in a semicircle is a right angle.
5. A triangle is determined if its base and base angles are given.

Of these the first and third are probably cases in which

8 James Gow, op. cit., pp. 140, 141.
Thales relied on intuition. The second is a case of experiment. The two remaining theorems are obviously incapable of such treatment, and must have been supported either by deduction or at least by very wide induction.

Cotiting from Archibald's *Outline of the History of Mathematics* concerning Thales:

Of practical problems he showed how to determine the distance of a ship from the shore and found the height of a pyramid by means of the shadows cast on the ground at the same moment by the pyramid and a stick; that moment was chosen when the length of the stick and its shadow are equal. There is good evidence that Thales predicted the solar eclipse which took place on 30 May 585 B.C. The basis of this prediction was probably the result of observation made by the Babylonians.

Thales' most noted pupil was Pythagoras who was with him for a short time and who was advised by him to continue his studies in Egypt. Pythagoras opened a school in Croton, a wealthy Greek city in southeastern Italy. Here he gathered about him several wealthy young men who had time for study and research and founded the first secret society known to Europe. Many advanced steps in the philosophy of mathematics and especially geometry were seriously studied here. They proved the following proposition: the plane about a point is filled by six equilateral triangles, four
squares or three regular hexagons; the sum of the exterior angles of a triangle is two right angles; the sum of the squares on the sides of the right-angled triangle equals the square on the hypotenuse, a fact known to the Egyptian "rope-stretchers", but first proved by the Pythagoreans.

An interesting idea may be quoted from Smith's *History of Mathematics*:

The Pythagorean Theorem is not uncommonly called the "pons asinorum" by modern French writers. The Arabs called it "Figure of the Bride", possibly because the Euclid figure is not unlike the chair which a slave carries on his back and in which the Eastern bride is sometimes transported to the ceremony. 10

The Pythagorean geometry, like the Egyptian, is concerned more than that of Thales with the relations of areas and volumes, particularly the five regular solids, and is not largely concerned with those relations of lines which do not readily suggest arithmetical expression. Pythagoras changed the study of geometry, for he examined its principles and investigated its theorems in an intellectual manner. As did other teachers of the time, Pythagoras taught orally because of the lack of suitable writing materials. No doubt, much of the actual work was lost, but his influence alarmed the civil rulers. He taught that the

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earth was a sphere and believed number to be the essence of all things in the universe. For these ideas he was exiled before his death and his brotherhood scattered, but their influence has lived on and came to flower in the golden age of Plato and Aristotle.

Many mathematicians of less spectacular fame followed Pythagoras and continued to add to geometric knowledge. Democritus, who lived about 400 B.C., is said to have been the first to demonstrate that the volume of a cone is exactly one-third of a cylinder of the same base and height and that a pyramid has the same relation to the prism of equal dimensions.

The philosopher Plato, although not a mathematician, had a great respect for the study of geometry. He was convinced that mathematics was of great value in disciplining the mind to logical thinking and it is said that the entrance to his Academy bore the inscription, "Let no one ignorant of geometry enter here". 11

Like Plato, Aristotle was important for his attitude toward mathematics rather than for his contributions to the subject itself. He is said to have encouraged his pupils at his school, the Lyceum, to collect material for a history

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of mathematics, the influence of Aristotle is one of the most interesting commentaries on human thought. As tutor to Alexander the Great, Aristotle was in a position to collect great quantities of scientific information relative to the countries conquered by Alexander. His summaries of this information, added to his own philosophy, were so authoritative that they became great reference works of the Middle Ages. Medieval scholars tended unfortunately to accept Aristotle's conclusions without question and it is very likely that their adherence to his doctrines resulted in the retarding of scientific inquiry.

The last great school among the Greeks was founded at Alexandria, Egypt, the city of Alexander the Great. Alexandria replaced Athens as the center of scientific thought, probably because the new rulers of Egypt provided funds which created the greatest university and the greatest library of the ancient world. The names of all the great scholars and mathematicians who were connected with this school could not be given in this study, but their work went far in advance of intuitive geometry. The cheapness of papyrus as compared with parchment made Alexandria the center of the book-copying industry of the

\[12\text{Ibid., p. 10.}\]
Mediterranean world. It was here that Euclid\textsuperscript{13} compiled his \textit{Elements} about 300 B.C. He has the distinction of being the only man to summarize all the mathematical knowledge of his time. According to a summary of \textit{The Thirteen Books of Euclid's Elements} given by Vera Sanford in her \textit{A Short History of Mathematics}\textsuperscript{14}, the \textit{Elements} were not concerned alone with geometry. Originally this great work was written in "books", or a series of parchment rolls. It was so perfectly organized and complete that practically no improvement was made in his methods until the past century. It was still used as a textbook as late as the nineteenth century. A synopsis of the book of \textit{Elements} is given by Fink in his \textit{Brief History of Mathematics}. It reads as follows:

The first book of the \textit{Elements} deals with the theory of triangles and quadrilaterals, the second book with the application of the Pythagorean theorem to a large number of constructions, really of arithmetic nature. The third book introduces circles, the fourth book inscribed and circumscribed polygons. Proportion explained by the aid of line-segments occupy the fifth book, and the sixth book finds their application to the proof of theorems involving the similarity of figures. The seventh, eighth, ninth, and tenth books have especially to do with the theory of numbers. These books contain respectively the measurement and division of numbers, the algorithm for determining the least common multiple and the greatest common divisor, prime numbers, geometric series, and incommensurable (irrational) numbers. Then follows stereometry: in the eleventh book the straight line, the plane, the prism; in the twelfth the discussion of the prism, pyramid,

\textsuperscript{13} Ibid., p. 10.

\textsuperscript{14} Ibid., p. 270-5.
cone, cylinder, sphere; and in the thirteenth, regular polygons with regular solids formed from them, the number of which Euclid gives definitely as five. Without detracting in the least from the glory due to Euclid for the composition of this imperishable work, it may be assumed that individual portions grew out of the well-grounded preparatory work of others. This is almost certainly true of the fifth book of which Eudoxus seems to have been the real author.\textsuperscript{15}

Eudoxus, one of the most brilliant mathematicians of antiquity, was born at Cnidos on the west coast of Asia Minor. He was an original genius second only to Archimedes. He was known as an astronomer, physician, and legislator, as well as geometer. Besides the work given in the fifth book of the Elements, he discovered the so-called "method of exhaustion" by means of which he gave the first rigorous proof of the results for the volume of a cone, and of a pyramid, and probably revealed that: (1) the areas of two circles are to one another as the squares of their diameters; and (2) the volumes of spheres are to each other as the triplicate ratio of their respective diameters.

It seems certain that Menaechmus\textsuperscript{16}, a pupil of Eudoxus and a contemporary of Plato, was the discoverer of the conic sections, parabola, ellipse, hyperbola, which were originally thought of as sections perpendicular to

\textsuperscript{15}Dr. Karl Fink, op. cit., Ch. IV, pp. 198, 199.
\textsuperscript{16}Raymond C. Archibald, op. cit., p. 27.
generators of right-angled, acute-angled, and obtuse-angled cones. He is said to have added three kinds of proportion to those introduced by Pythagoras, and increased by the analytical method the learning begun by Plato, on the subject of the "section". This must mean what we call the "Golden Section", or the cutting of a line into extreme and mean ratio. This discovery of Menelaus was of particular interest because it gave a new solution of the problem of the duplication of the cube, one of the three famous problems which had been formulated in the fifth century, B.C. Archibald lists these problems:

(a) To find a line which shall be the edge of a cube whose volume is double that of a given cube, the problem of the duplication of the cube.

(b) To trisect any given angle.

(c) To find a line which shall be the side of a square whose area shall be exactly equal to that of a given circle, the problem of squaring a circle.¹⁷

All of these problems were solved by the Greeks within a century, but more than twenty-two centuries were passed before it was finally proved that no one of them could be solved by ruler and compass alone.

Another great scholar of Alexandria was Eratosthenes, the librarian of the University of Alexandria. He was the

¹⁷Ibid., p. 17.
first person to calculate the circumference of the earth. As Effie McDougall says in her thesis:

He did this remarkable feat by computing the arc of the circumference between Alexandria and the present site of Assuan dam both of which are on the same meridian. This he did by observing the sun and finding the exact difference in time between the two places. Thus he was able to learn the approximate length of a degree on the earth's surface and compute its circumference and diameter. He estimated the diameter within fifty miles of the present measurement.13

Eratosthenes also invented an instrument for duplicating the cube, and he calculated the distance from the earth to the sun and to the moon. It is said that he became a victim of one of the diseases of the eye and that when he was unable to read, he committed suicide.

Probably the greatest of all the great Alexandrian scholars was Archimedes.19 He was born in Sicily about 287 B.C., and was killed by the Romans in 212 B.C. Up to the time of Newton, he was the greatest mathematical genius that the world has ever seen. Quoting from Vera Sanford:

According to legend, Archimedes was noted among his contemporaries for the system of pulleys by

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19 For the life and work of Archimedes see Sir Thomas Little Heath, Archimedes, Cambridge, England, 1897.
which he drew a ship up onto the shore single-handed, for the burning glasses by which he set fire to enemy vessels in the harbor, for the engines of war by which he terrified even the Romans, and for his boast that, given a place to stand, he could move the earth. Among mathematicians, Archimedes is known for his work in mechanics, for his studies in the theory of numbers, and for investigations which foreshadowed the calculus.

To his friend Eratosthenes he dedicated his treatise on Methodus. Many years later Cicero discovered the tomb of Archimedes and said that it bore the figures of a cylinder and a sphere, which were symbols of his investigations.

Apollonius was the last of the great Alexandrian scholars and was twenty-five years younger than Archimedes. His contemporaries called him the "Great Geometer" because of his extraordinary treatise on conics and also mentioned him as a famous astronomer. As in the name of Euclid, so in the name of Apollonius of Perga, there has been controversy. The theory of curves of the second order does not begin with Apollonius any more than does Euclidean geometry begin with Euclid. To Apollonius are due the terms parabola, ellipse, hyperbola, and to him is also due the first textbook upon the subject. This work consisted of eight books, the first four of which have been preserved to us in Greek manuscripts. The fifth, sixth, and seventh were translated into Arabic, from which source they were carried

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to the West. The eighth book was lost. With Apollonius, the period of new discoveries in the realm of the theory of conics comes to an end. After his day the light of knowledge gradually grew dimmer and dimmer until Europe, for a thousand years, groped in the darkness of profound ignorance.

The century which produced Euclid, Archimedes, and Apollonius was the period in which Greek mathematical genius attained its highest development. For many centuries afterwards geometry remained a favorite study, but no substantive work was ever produced which could be compared with the "sphere and cylinder" or the "conics". Two great divisions of geometry may be designated by the names "Geometry of Measurements" or "Geometry of Archimedes", and "Geometry of Form and Situations" or "Geometry by Apollonius". The geometers who succeeded these are professors who signalized themselves by some commentary on the classical treatises but by very little mathematical discovery.

The materials for a history of Greek geometry after Apollonius are scarce in quantity and most unsatisfactory in quality. Frequently what was inherited from the Greeks was not even fully understood, and therefore remained buried in the literature of the foreign nations. From the time of the Renaissance, however, and especially from that
of Descartes, an entirely new era with more powerful resources investigated these ancient treasures and made them a part of our present culture.

There are names of many geometers who lived during the next three centuries, but few have come down to us, and we are compelled to rely for the most part on such scraps of information as the later scholars Pappus, Proclus, Eutocius, and others have incidentally preserved. During the whole period from Apollonius and Ptolemy there are only two geometers of real genius, Hipparchus and Heron. From what information one can gather, they both lived about 150 B.C. Hipparchus was really an astronomer, while Heron was above all things a surveyor and engineer.

About 70 B.C. lived Semaus who seemed to have been the freedman of a wealthy Roman and who wrote, besides the astronomical work, a book on the Arrangement of Mathematics, which without being expressly historical contained abundant notices of the early history of Greek mathematics. A book of this kind written not long after the classical age, if preserved, would clear up many difficulties which do not now admit of solution. The Romans were in general uninterested in the speculative side of the Greek mathematics. In fact, Cicero congratulated his countrymen because they concerned themselves only with the mathematics needed in measuring and reckoning. It is
true that the Romans used a certain amount of mathematics in building their aqueducts, but they contributed nothing to the theory of mathematics. They were obliged to employ specialists from Alexandria when Agrippa (63-12 B.C.) carried out Julius Caesar's design of making a survey of the Empire.

Although Hindu geometry is dependent upon the Greek, yet it has its own peculiarities due to the arithmetical modes of thought of the people. Later on they were influenced by the Chinese but most of their original works concerned arithmetic. The Arabs did not succeed in a single point, even in the subject of conic sections, beyond what had been reached in the Golden Age of Greek geometry. Within a century after the death of Mohammed (632), his followers conquered Egypt, Northern Africa, and a part of Spain. They maintained their foothold in Europe until the fifteenth century. In the East, Arab conquests reached to India. During this period Arab culture dominated the Mediterranean world and Europeans reached out for this knowledge. Vera Sanford referred to this as "The Period of Transmission".21 Quoting from her book A Short History of Mathematics:

The Greek classics found their way into Europe

21 Ibid., p. 21.
by several different routes: through the Norman kingdom of Sicily, through direct contact with the Arab civilization in northern Africa, and through the Moorish universities in Spain; of these, the work of the Arabs is of particular interest although the other contacts were of great importance also...  

One of the first Christians to study in the Moorish universities was Gerbert (960-1003), who later became Pope Sylvester II... He wrote on arithmetic and on astrology, drawing his material from Arabic sources. His geometry, however, was based on manuscripts describing the methods of the Roman surveyors, which he found in the monastery in Italy.  

The Arabs were equally rapid in their assimilation of Greek and Hindu science and mathematics. Their religious beliefs and practices made necessary the precise fixing of their great feast days by the phases of the moon and even entailed the accurate determination of the hours of the day. Naturally the majority of Arab mathematicians began their careers as astronomers. The learning of the Greeks passed over in the ninth century to the Arabs and with them came round into the west of Europe. The Arabs were chiefly interested in arithmetic, trigonometry, and algebra instead of geometry. In the sixteenth century Greek geometry again became known in the original and was studied intensely for about one hundred years until Descartes, Leibnitz, and Newton, the best of its scholars, superseded it.

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22 Ibid., p. 21.
During the period 500-1300 the student went to the
teacher in the monastery and heard his lectures. In the
which burned down both the Schools of Salerno and
thirteenth century, however, universities commenced to
spring up at such places as Bologna, Padua, Naples, Paris,
Oxford, and Cambridge. Scribes were kept very busy by
the universities making copies of treatises. By the middle
of the fifteenth century their products were being sold as
books as today.

Italy made its chief contributions to mathematics
through Leonardo of Pisa. About 1220-1229 he wrote two
important works: Liber quadratorum and Practica Geome-
triae. The latter brings together a vast amount of mater-
ial in geometry and trigonometry and it would seem as if
some works of the ancients now lost had been available to
Leonardo. The amount of mathematics taught in the uni-
versons was very meager, but the amount known by the
merchant class even in Italy, which was far in advance of
the rest of the world, was limited to casting accounts by
means of counters and other simple operations. Bookkeep-
ing was managed by the primitive method of the tally
sticks. In a booklet from the Burroughs Adding Machine
Company on The Story of Figures we find:

Up to 1645, the British Government kept
records of transactions by the tally system.
After the system ceased, the basement of the
House of Commons remained cluttered with vast
accumulations of these dry sticks for nearly two centuries. Finally, it was decided to burn them. The stove became overheated and a fire ensued which burned down both the House of Commons and the adjacent House of Lords. 23

The next outstanding geometer was the Frenchman, René Descartes (1596-1650), also a philosopher and physicist. He abandoned the publication of a work dealing with the physical theory of the universe, feeling that it would arouse the antagonism of the Church, but in 1637 he published a treatise on universal science called Discours de la Méthode. His Géométrie, which was concerned both with analytic geometry and the theory of equations, was an appendix to this work. 24

The seventeenth century is especially outstanding in the history of geometry. It saw Descartes invent analysis, Pascal and Desargues open up new fields for pure geometry, Galileo Galilei reveal laws of mechanics, Newton create new worlds with calculus, curves, and physical observations, and Leibnitz build master notations. An interesting comment concerning these men was made by J. T. Borer in an article "The Social Qualities of Mathematics". It reads as follows:

History is one of our chief social studies. . . . What part has counting played in the uplift of mankind? What economics part has algebra played? . . . What did Napier and Briggs do for us? The great Newton, Leibnitz, Descartes and a host of other discoverers have done more for us than Julius Caesar and Napoleon Bonaparte. 25

Galileo was the founder of the science of dynamics. A native of Pisa, he carried on experiments from the leaning tower. He speculated interestingly on inertia. He constructed telescopes in 1609. This invention led him to discard the old theories of the solar system and he finally was tried for heresy on the charge that he had said that the earth moved around the sun. Less is known of Galileo's pupil, Cavalieri (1598-1647). His work was mostly with integral calculus.

Isaac Newton (1642-1727), at Cambridge University, had invented at the age of twenty-three the method of calculus, and discovered the binomial theorem. In the following years he made applications of the calculus to tangents curvature, concavity and convexity, and the law of universal gravitation. His greatest work was the Mathematical

Principles of Natural Philosophy published in 1687.

Leibnitz was the only pure mathematician of the

first class produced by Germany during the seventeenth century. He made important contributions to the notation of mathematics. Not only is our notation of the differential and integral calculus accredited to him, but our signs in geometry for similar and for congruent.

By the year 1700, mathematics was ordinarily taught in the elementary and secondary schools, but relatively little had been added to this body of subject matter in the last two centuries. The English mathematicians of this period were less prominent, being overshadowed by the giants of the preceding era. Following the death of Newton, the principal concern of mathematicians in England was the disproving of the claim of Leibnitz as inventor of the calculus, and the creation of a rigorous basis for the subject. The publicity given to the controversy caused much popular interest in mathematics. It was not unlike the situation a few years ago when laymen read popular works which purported to explain the Einstein theory.

It seems proper to mention Leonard Euler (1707-1783), the great Swiss mathematical genius. The project of creating an Academy at Peters burg, which had been founded by Peter the Great, led to Euler's being invited in 1735 to the chair of mathematics there. Few writers in mathematics ever contributed so extensively or so fruitfully. He is
considered the creator of modern mathematical expression. He revised almost all the branches of mathematics then known, filling up details, adding proofs and new results, and arranging the whole in a consistent form. One of the theorems of this "sequel to Euclid" was proved by Euler (1765) when he showed that the intersection of the altitudes, the bisectors of the angles, and the medians of a triangle are collinear.

The reader will note that practically all mathematics used by the American engineer was developed before the nineteenth century. The latter part of the eighteenth century witnessed a revival of interest in pure geometry which has had important results in the development of new subject matter. Although the many phases of modern geometry are beyond the scope of high school work, it is desirable to mention certain outstanding developments, unless the reader should conclude that progress in recent times has been confined to the analytic and non-Euclidean geometries.

Introductory work in descriptive geometry had been done at an earlier date, but Gaspard Monge (1746-1818) developed it independently and carried his studies far beyond the point which others had previously reached. Descriptive geometry was of great importance to the govern-
ment in the science of fortification, and accordingly Monge was not allowed to publish his researches at the time they were first made. About thirty years later this ban was removed and his Geometrie descriptive, based on lectures given at the Ecole Polytechnique, was published in 1799.

Gerard Desargues (1593-1622), though really belonging to the previous century, is mentioned here for his work was not really appreciated until the nineteenth century, when interest in pure geometry was reawakened. Desargues was interested in the study of perspective and his researches in geometry led him to the use of the line at infinity, poles and polars, and other such topics. His work was overshadowed by that of Descartes, although it was along quite different lines. Fundamental ideas of projective geometry were used by Pascal in his study of cones and by Newton in his work with cubics.

The first half of the nineteenth century was a period of rapid progress for projective geometry. Among the important contributors to the theory, as listed by Vera Sanford, were Carnot, Poncelet, Cergonne, von Staudt, and Steiner. Carnot (1783-1823) was a pupil of Monge. He

\[26\] Vera Sanford, op. cit., pp. 226, 229.
introduced the concepts of a general quadrilateral and of negative magnitudes. Another pupil of Monge was Jean-Victor Poncelet (1788-1867) who developed the idea of continuity, and worked extensively with anharmonic ratios and the circular points at infinity. He was taken prisoner when Napoleon's army retreated at Moscow in 1813. Although deprived of books or assistance of any kind, he devoted his period of imprisonment to the study of projective geometry. He published this treatise in 1822 under the title Traité des propriétés des figures.

Joseph-Denis Gergonne (1771-1859) developed the subject of poles and polars and was the first to discuss the "class" of a curve.

Karl Georg Christian von Stämm (1799-1867) used only the properties in his Geometrie der Lage (1847), showing how analytic methods could be introduced without the idea of measurement.

Jacob Steiner (1796-1863) was an illiterate boy of fourteen when Pestalozzi became interested in him. Eight years later he entered Heidelberg and later became a professor at Berlin. His researches in geometry extended Carnot's work on the quadrilateral to a polygon in space and also treated curves and surfaces of the second degree.

The publication of Euclid's Elements about 300 B.C. served in a way to close the field of geometry to further
work and little was added until the renaissance of pure
gometry in the seventeenth century when Descartes and his
contemporaries developed the subject of analytics and when
Fuscal and Newton were paving the way for the modern geo-
metry of recent years. The last two centuries have been
noteworthy for the development of these various branches
of geometry and for the critical study of its concepts and
postulates.

When geometry outgrew its intuitive stage, it became
the greatest of all mathematical sciences in the ancient
world. It suggested the related science of trigonometry
and together they made possible our modern knowledge of
astronomy, our present methods of surveying, and our modern
designing of machinery or mechanics. The Greeks expressed
their geometric ideas in beautiful architectures, marble
statuary, and the fine arts. Modern Americans express
their mathematics in wheels and cogs of steel which have
created industries and the mass production of goods. Yester-
day only the scholars were affected by the study of
mathematics, but today everyone, young or old, rich or poor,
is directly affected by mathematics and its innumerable
applications.
CHAPTER III

FUNCTIONAL TRENDS IN PRESENT-DAY GEOMETRY TEACHING

As a scientifically organized division of mathematics, geometry is the oldest of its branches. This was emphasized in the previous chapter on the history of geometry. For this reason it has had a longer period in which to perfect itself and is, therefore, looked upon as less capable of reform or improvement.

The last quarter of a century has shown, however, that as a school subject it is capable of improvement in the same manner, if not to the same extent, as are the other branches of mathematics.

In recent years many writers have clearly differentiated between intuitive and demonstrative geometry. In 1912 at the Cambridge meeting of the International Mathematical Congress, intuitive geometry was brought prominently before the educational section of that organization. Accordingly, it began to be seriously considered by groups of teachers throughout the world. Since then intuitive geometry has
come to this country and occupies a worthy place in many of our courses for the junior high school. This subject naturally precedes demonstrative geometry and our schools have come generally to recognize that it has but little sanction in the more mature branch of mathematics.

The progress of geometry in our senior high schools has been steady and encouraging. David Eugene Smith briefly summarizes the nature of intuitive and demonstrative geometry as follows:

(1) There has been a more definite recognition by the schools that the chief purpose of demonstrative geometry is to show the application of logic to the proof of mathematical statements. It requires a maturity of mind hardly found before the tenth school year, although for purposes of information a little work in demonstration may properly be given to the able pupils in the preceding grade.

(2) Therefore the purpose of demonstrative geometry is not mensuration, this being sufficiently cared for in the work in intuitive geometry; its purpose is, in part, to demonstrate the truths already known intuitively. For this reason the work in the mensuration of the circle has little sanction in demonstrative geometry, the rules being already known from intuitive geometry and the demonstrations as given not being very satisfactory from the standpoint of logic. The subject is, therefore, no longer required in college entrance examinations or for high school graduation. The same is true as to the mensuration of the rectangle, the rectangular solid, the cylinder, the sphere, and the cone.

There are four values of the study of geometry recog-

nized or claimed, according to William L. Wrinkle in his *The New High School in the Making*, namely, cultural, preparatory, functional, and disciplinary.\(^2\)

The cultural values, which include two phases, have not been clearly defined by mathematicians. In the first place, we need to know geometry for general information, because the rest of the world knows something of it. This is not a very determining reason, but it is one which would justify keeping any traditional subject in the curriculum. The other important cultural phase is that of logic of geometry. Many propositions were known before Euclid, but he was the one who arranged them so that they are one of the most admired collective arrangements in logic that have ever been produced. This logic has given added significance and beauty to the truths themselves.

The preparation value of the study of mathematics, expressed in the terms of units of credit demanded by many colleges and universities for entrance, represents without question the chief justification for the prominence of mathematics in the senior high school program. Usually two years of mathematics are required, one of algebra and one

of geometry. If the student is planning on an engineering course, he must also have solid geometry.

According to modern educational philosophy, as represented by Dewey and his followers, the only wholly defensible basis on which mathematics may build its claim to a place in the secondary school as a required subject is in terms of its functional contribution to everyday living. From the functional point of view, the program in the secondary schools should include such branches of mathematics as may be needed by the individual in the solution of his problems or for the satisfaction of his interests. Mathematics, of course, is involved in many everyday life situations.

An application of the idea that mathematics instruction should be concerned only with functional values would undoubtedly scrap much of the material now included in mathematics courses of study and textbooks, but it would bring into the curriculum many problems which are untouched by the present conventional program. The second part of this study deals with functional problems that may be used to supplement the adopted textbooks in teaching solid geometry. These problems can be used as assigned work, extra ("enrichment") work, or for testing fundamental skills taught in class. The most progressive courses are now beginning to recognize the need for functional material to
supplement present day textbooks, but up to this time such material is not available in a form for classroom use.

During the past quarter of a century many, if not most, educational psychologists have attacked the idea of wholesale disciplinary values to be derived from studying geometry. From Plato's day to the present, one of the main reasons of teaching mathematics has been its disciplinary value. The early extreme position, known as faculty psychology, maintained that large amounts of transfer were easily made; but not so today. William James\(^3\) thought that the idea of transfer should be subjected to experimentation. In 1890 his experiment showed little or no transfer in memorizing. Other experiments were made by Gilbert and Freacher in 1897\(^3\), Thorndike and Woodworth\(^3\) were also among the pioneers, with experiments conducted in 1901. In all, some thirty experiments of importance reported the amount of transfer had been low, but it is probable that these investigations failed to reveal any large fractional part of the full transfer. The experimental settings have not only been far from perfect, but nearly all of them have been somewhat artificial in charac-

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ter. As a result, school people and casual readers have actually come to believe that there is no transfer and that, therefore, each specific skill and item of information has to be taught in its natural setting. Many writers nowadays who use such phrases as "now that the idea of transfer has been exploded" (often they had been exploded) are obviously not familiar with recent investigations of Lashley, Crats, Overman* and others which are driving sensible people to the middle of the road. The two extreme views for and against disciplinary values practically no longer exist among well-informed school people. We shall almost certainly have a revival of a modified theory of transfer of training.

The acceptance of this doctrine does not mean that pupils should be taught disagreeable and difficult tasks. If the procedures used in reflective thinking are to be mastered, how much more sensible it is to choose for a given individual tasks which are not only interesting to him but useful.

The extent to which mathematics is interwoven in modern life was vividly displayed in the Century of Progress in Chicago. The display in the Hall of Science in-

cluded a great many fascinating projects that our average citizen might wish to understand. The mathematical needs of common life are largely concerned with size, shape, position, and of the relationship of things to each other. This is the foundation of rational thinking concerning which Cassius Keyser has said:

Each thing in the world has named or unnamed relations to everything else. Relations are infinite in number and in kind. To be is to be related. It is evident that the understanding of relations is a major concern for all, men and women. Are relations a concern of mathematics? They are so much its concern that mathematics is sometimes defined to be the science of relations. There are increasing numbers of people who feel that mathematics should be taught as a means of gaining an objective basis, a fact-finding spirit, and a quantitative procedure in dealing with the perplexing social and economic problems of today. The proposal that these be clarified is at least worthy of careful consideration.

Johanna E. Herbart\(^6\) conceived the learning process as "synthesis". He regarded it as a matter of putting exercises together into system of related ideas. Attentiveness to form is obviously a mark of maturity from this point of

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view. Mathematics sharpens apprehension and anticipates the next stage of learning by bringing the attention to bear upon the formed and quantitative aspects of the immediate experience. This obviously does not apply to mathematics as presented in complete detachment from the learner’s developing experience. In order to perform this function it must be very closely connected with the immediate interests.

J. H. Blackhurst, in his article in *The Mathematical Teacher* entitled “The Educational Value of Logical Geometry” says:

> Deny moved beyond Herbert leaving faculty discipline still farther beyond until at the present time faculty discipline is no longer scoffed in theory; it has been all but forgotten. Few educators could tell exactly what it is, or was thought to be. In spite of this fact we are still slaves to its clutches as far as the teaching of logical geometry is concerned.

> Meanwhile educational theory has swung to the opposite extreme and is emphasizing education as a process of learning to live by living.”

> While practice is still under the influence of faculty discipline, educational theory has gone ahead ignoring mental discipline except that which is incidental to life-centered situations. There is a need, however, for a study

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of the scientific method of induction as such and also a need for a study of the processes of deduction. Such a discipline in reasoning might be called a discipline of processes in contrast to faculty discipline.

Probably the next step in education will emphasize discipline as a means of putting stiffening into "life-centered" education. If this happens geometry will be just the subject with content which will lend itself to a study of the processes of reasoning. In other words, there is a compelling need in education which can adequately be met by the subject of geometry. But to meet this need geometry must have its content so organized and taught as to make it a source for exhibiting and studying the processes of rigorous reasoning.

Many people have been too slow to see the educational possibilities of geometry, with the result that there is danger that geometry might pass out of education as did formal grammar. If it does, it would take more than a century of time to bring it back. The tragedy of such an event lies in the fact that geometry best meets the need for experience in thinking. The materials in geometry might be used to introduce the pupil to a formal study of scientific method, the method of induction, as well as a formal study of the deductive method.
The pattern of thinking is clearly illustrated in geometry and in non-geometric situations by Dr. H. C. Christofferson's lecture before the National Council of Teachers of Mathematics when they met in Indianapolis, December, 1937. He uses the geometric theorem, "if a triangle is isosceles the angles opposite the equal sides are equal", and a theorem in non-geometric thinking, "if all standing water is covered with a film of oil then there will be no more mosquitoes". His chart endeavors to picture the dependence of proof upon definition, previous theorems, and postulates. Clear, accurate, precise thinking in any situation will involve many undefined terms and some definitions, several statements accepted without proof like postulates, and some statements for which proof will be necessary. In this he used inductive and deductive reasoning, as well as direct and indirect proof.

The latter part of his paper deals with some phases of good thinking which will be summarized briefly. He listed some commonly used terms which need careful definitio-

8 Dr. H. C. Christofferson, "Geometry a Way of Thinking", The Mathematics Teacher, XXXI, April 1938, pp. 147-154.
9 Ibid., 149.
"Grease the car" is a common phrase which seems to have a special meaning. What part do you want greased - upholstery or wheels? Just what do you mean? "No U-turns", "one-way", "slow", "stop school bus", and many other phrases which will be recognized by any motorist as needing definitions. Meaning is quite different in many cases from the apparent one. When applied to the stock market "bull" and "bear" have technical meanings. One must know these meanings before he can understand what is said or written. The same is true for such words as capital, days of grace, C.O.D., net, par, and others.

There are many systems of postulates outside geometry. One's life and actions are governed by the postulates in which he believes; for example, "Honesty is the best policy". Our Declaration of Independence suggests a system of governmental policies. It makes a person realize how important such principles are when one considers what they mean to such nations as Japan, Italy, Germany, and Russia. Each country has a different set of postulates which they accept without proof as the basis for their government. All their reasonings about war, conquest, individual rights, and other governmental affairs are determined by the postulates of their country. Rules of a game are really postulates. They are used to prove
which tens win points and the game. Laws of science such as "law of gravitation" and "the law of levers" are illustrations of accepted statements which seem to account for certain results which have been used as postulates.

Geometry is really a deductive science, yet much thinking is inductive in nature at first. Then conclusions are proved by rigorous deductive reasoning. This can be more easily understood by Dr. Christofferson's explanation. He uses the theorem "if two angles of a triangle are equal the sides opposite them are equal". He used fifty degrees for each of the base angles but he proved his conclusion by using only previously proved or accepted statements which is the deductive method. Such proof does not depend upon the size of angles in degrees. Neither does the language spoken, nor the climate influence the result. Deductive reasoning is not handicapped by instruments of observation or measure. It is exact, accurate, and unquestionable in or out of geometry. This is spoken of as being mathematically rigorous. The meaning of mathematical rigor is that the conclusion is the correct and only one which can follow from the hypothesis, the postu-

\[\text{\textsuperscript{10} Ibid., 151.}\]
lates, and the other theorems upon which it is based. 

Much of our so-called thinking is not rigorous but purely inductive. For an example, June has red hair and is jolly, John has red hair and is jolly; therefore, all red-haired people are jolly. Much modern advertising depends for its efficiency upon our non-rigorous inductive thinking. A picture of a beautiful girl smoking a certain brand of cigarette is suggestive to girls and boys that they too smoke that kind of cigarette. Many of the conclusions of science are at first purely inductive in nature. For example, a certain type of treatment cured eighty out of one hundred cases; therefore, it is a good treatment. Yet much of the work of science and engineering cannot take chances. Engineers will use certain laws or postulates which have been discovered concerning stress and strain to build a bridge. These laws have been discovered by inductive reasoning and established by experiment. Therefore, it may be said science is built upon laws experimentally determined, really postulates based on inductive thinking, then, accepting these as true, rigorous deductive reasoning is done.

Deductive thinking in a non-geometric situation was illustrated earlier in this chapter. While deductive reasoning in non-geometric problems resembles in many ways
the same reasoning in geometry, there are striking differences. The dependence of the conclusion upon the premise and upon other conclusions which may be thought of as postulates or previous theorems is exactly the same. The chief difference lies in the larger number and the complexity of both postulates and previous theorems, as was shown in Dr. Christofferson’s illustration of the mosquitoes controlled by oil on all water.

Converses or opposites in non-geometric reasoning are very common. Frequently they are assumed to be true without proof, but all converses need proof just as they do in geometry. Too often teachers never mention converses which are not true. Therefore, some students are misled by thinking all converses are true. In geometry, however, every converse is proved. Further information of this would be interesting, but in this study we will only mention it. In fact, one common form of indirect proof uses the converse statement in the process of exhausting all the possibilities. Another form, probably the most useful in proving a statement true, is by proving its opposite false. Indirect reasoning is commonly used by each of us in finding certain books in the library, locating a student in a large school, in repairing anything.

11Ibid., p. 148.
in planning a meal, in deciding what club to use in golf, and in many daily tasks.

Careful thinking in any field of activity involves definition of terms used, and very likely some postulates and previously proved conclusions, and even converses or opposites. Generalizing from a few cases (inductive) is far too frequently used and too seldom do we use rigorous deductive thinking.

This chapter has related the more recent trends concerning geometry. Since intuitive geometry has come to this country, it has found a worthy place in many of our courses in junior high school. The demonstrative geometry is taught in senior high school with its chief purpose to show the application of logic to the proofs of mathematical statements, and to demonstrate the truths already known intuitively. Four well-recognized values of the study of geometry are the cultural, the preparatory, the disciplinary, and the functional. The latter has been added recently and if properly taught will call for new books and new supplementary materials. Such a manual will be found for teaching solid geometry in Part B of this study. By proper use of this manual, geometry may be applied as a way of thinking to the many geometric and non-geometric situations of life. There are many problems of
the present curriculum to be solved. Ideas concerning transfer of training are still questioned. Many recent articles discuss ways of vitalizing geometry and new methods and techniques used. These will be discussed in the following chapter with the hope that they will reveal further need for the supplementary manual found in Part B.
CHAPTER IV

PROBLEMS IN TEACHING GEOMETRY

By 1912, geometry was required for entrance in every American college. With the production of suitable textbooks, work in elementary mathematics has made its way from the university into the secondary schools until it is generally required for college entrance work, instead of as a subject for the average college student. There is no reason for expecting this shifting has reached its end. A considerable change may be anticipated as the result of the demand of modern science and economics that the student be prepared to make constant use of the tools and concepts of mathematics.

Euclid paid little attention to solid geometry, with the result that his followers in the English schools have also neglected it. Since the conservative Eastern states have always been influenced by the educational traditions of England, solid geometry has never had the hold in the preparatory schools that it has in the Central
and Western states, for here tradition counts for less. Of course, there were arguments for both sides. On the one side, there was not enough time to teach plane, to say nothing of the solid. The other argument was that the whole question should be left to the teacher. She had a year at her disposal for geometry. If she desired she could devote one-third of the time to solid, since it would add to the students' interest and would contribute to the practical side through the knowledge of mensuration. An advisory committee in the Los Angeles City School District made up of teachers and supervisors for elementary through high school made suggestions which concern this study as follows:

Many feel that formal mathematics in the secondary school should be considered as largely vocational in that it may be basic to advanced study in professional or technical fields. Even from this standpoint there may be a differentiation in type of content. Those who may specialize in engineering design or in the professional field of mathematics will need the most rigorous type of specialized training . . .

Those who will need advanced mathematics constitute perhaps a relatively small proportion of our pupil population, but they are an important group. While the entire mathematics program would not be dictated by their needs, they should receive adequate consideration. The unfortunate reality about mathematical power is that it is attained by means of steady growth over a relatively long period of time and cannot be acquitted quickly to meet current needs.

This group of individuals desiring courses which require solid geometry is a very important group. From this group emerge those highly specialized individuals who advance the science and mathematics of the future. They deserve the sympathy, encouragement and help of our best teachers. "A select group of pupils for the entire high school is no longer possible," Karl R. Douglas relates a few facts to us as follows:

In 1900, 27.4% of all high school pupils were studying geometry. This means that practically all students studied geometry in either their junior or senior year. In 1935, less than 10% of the pupils were studying either plane or solid geometry. . . . This is attributed to the tendency to make secondary education universal. It is of great significance to those interested in the teaching of mathematics that high school enrollments have just about reached the saturation point."

Previously he gave the estimated enrollment in public secondary schools in 1900 as 519,000 and in 1935 as 6,100,000.

W. D. Reeves in discussing modern curriculum problems comments as follows:

The high school population has increased more than 1300% in the past thirty-nine years. However, the high school still approximates the best fifty per cent of American intelligence.

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Just imagine how some of our teaching problems would be complicated if everybody were in school.3

When the public school was considered a selective institution for preparing a few superior children for the college education the mathematics courses may well have served as an intellectual hurdle. Today when the general public believes that a high-school education is the right of every child, there will be little patience with the curriculum which requires one or two subjects for all, especially where failure in these subjects is presumed to be an adequate measure of incompetence.

Many children are less willing to work for work's sake than their parents. Their brains are no less active and keen, but many students are much less prone to accept the dictates of others and are given to questioning the values of any subject.

What is equally important is the growth of knowledge in other significant fields, such as general psychology, social psychology, and sociology, which mathematical educators cannot ignore. Along with this has grown awareness of the fact that this generation and many generations to come face a rapidly changing economic world which calls

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for new patterns of quantitative thinking. All these factors call for a thorough reexamination on the part of mathematics teachers and the role of mathematics in the modern curriculum.

Quantitative relationship exists throughout the vast realm of human relationships in mathematical thinking, as well as in the more limited area of the exact mathematical sciences represented by algebra, geometry, and others. Many teachers are not successful in conveying to their students a capacity to recognize quantitative relationship and quantitative concepts when they are encountered outside the classroom. According to Willard W. Beatty, "New materials must be found which relate mathematical concepts to experiences or advanced mathematics will follow Greek out of the curriculum".4

It is a recognized fact that students are interested in experiences in which they themselves have shared and seldom interested in the accumulation of knowledge which cannot be connected in some way with these experiences. Almost any normal boy is interested in swimming and doubtless many times has watched with interest the

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pool being filled. Probably, too he has speculated on how much water it would take to fill the pool. An alert teacher can capitalize this interest and the usefulness of geometry will at once become vitally apparent to boys estimating the capacity of their own pool. This problem in actuality is found in the manual accompanying these pages. [Unit II, sec. D, ex. 9.]

Again Bill’s father may be surprised and delighted to learn that his son can actually estimate the cost of a double coat of paint for his flat-roofed garage. Bill has initiated the project and at the end has carried it successfully to a conclusion, has probably had the cooperation and direction of his gang in its accomplishment and evaluated his own work and found it good. Again Bill has realized that geometry is alive. [Unit II, sec. B, ex. 1.]

Monotony is almost unavoidable in the teaching of any subject unless a diversity of materials is definitely sought. In solid geometry functional problems motivate the students because they provide something vital with human appeal. Critical thinking and evaluation come from having actual problems to think about and solve. Even students in advanced geometry vary greatly in ability, interests, and background. Certain forms of manual expression should predominate with some students and oral expression with others. The exercises in Part B of this
study will furnish an excellent review of fundamentals. The method for using such supplementary material depends to a large extent upon the type of learning desired. There is no one best method for all teachers. Most methods have their good points when used at appropriate times and by teachers who know how to use them efficiently. Teacher training and interests have much to do with the effectiveness with which they use such exercises. The extent to which supplementary materials are actually used in the classroom depends at present very largely on the background, inclination, and insight of the individual teacher.

The following is an interesting comment by Idaila Waters from an article "Vitalizing Geometry Through Illustrative Materials";

There is practical agreement that the main justification of demonstrative geometry relates to logical thinking, at least so far as this is confined to mathematical thinking. According to the Gestalt psychology, if we wish to understand an idea we cannot do so until we see it in the proper relationship to the culture of which it is a part. It is the study of ideas or things in relationship which gives meaning. Hence, we need to bring into our classroom applications from everyday life if the pupil is to have a better understanding of the theorems which have been studied from the time of Euclid to the present day - in other words bring in
illustrative material in order to give more meaning to what we have always studied.\(^5\)

Our textbooks have been patterned too closely after the model of Euclid, whose text was not intended for use by boys and girls but for mature men. Our textbooks contain many more illustrations and much more practical material than formerly. But after all the greatest responsibility lies with the teacher. She must inspire interest and enthusiasm for her field. She must bring geometry down out of the clouds of abstractness.

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CHAPTER V

CONCLUSIONS AND RECOMMENDATIONS

The debate on the amount of transfer which was mentioned in the preceding chapter has had a wholesome effect on the teaching of geometry. The teacher of geometry now realizes that the amount of transfer probably depends on how much practical application is made in teaching the subject. Similarly, she realizes that she must plan for transfer and that the student must be aware of the attitudes and habits which create greater transfer. Since general transfer of teaching has been severely criticized, many instructors are substituting more specific problems to meet the learning needs.

If a student is to gain any benefit from geometry he must be convinced that the material of instruction is worthwhile. Materials can be brought into the classroom which will make the pupil realize that an automobile, a bridge, and a skyscraper represent applied geometry. There are many trades which depend for their existence
on blueprints and scale drawings. The contractor, the
builder, and the mason must appeal to geometry to avoid
waste of material. It is by geometry that the surveyor
makes his maps, the civil engineer his railways, and the
sailor navigates the ocean with success. The fine arts,
such as designing, making of pottery and jewelry, and the
use of art in advertising, all have a geometric founda-
tion. Numerous industries such as those which produce
clothing, furniture, and wall-paper would be crippled
without the knowledge of geometric methods in drawing.
Even the large per cent who are not going to college
will employ geometry in ordinary carpentry, ditch-digging,
excavating, house-painting, and paper-hanging. If all
this could be brought into the classroom, then many pur-
poses could be accomplished at once. Traditional sub-
ject matter could be better motivated and the student
could be given an opportunity to gain an interest which
might develop either into a hobby for leisure time, or
an occupation.

Many critics have pointed out that textbook pro-
blems are often artificial and that too frequently the
situation is far from real. Examination of solid geom-
etry books published during the last ten years reveals
unmistakably that definite progress is being made in the
selection and organization of geometric materials. These changes are not expected to take place rapidly, on account of the well-known conservatism of high school teachers.

This is a technical age and the materials appearing in magazine articles, radio broadcasts, and movies require more and more an understanding of geometrical concepts for a full appreciation. Many problems of life are not as conveniently simplified as the typical problem in a textbook. To hold the attention and stimulate enthusiasm in the student, one must plan so as to have a variety of problems. Such exercises are found in Part B of this study. Many of these exercises are practical problems which concern real life situations. They help link the textbook material with present day living and broaden the view of solid geometry beyond the mere manipulation of numbers, formulae, and abstract figures. They are psychological as well as logical. Functional problems as are found in the manual motivate students because they provide something vital with human appeal. Critical thinking and evaluation come from having actual problems to think about and to solve. Such exercises furnish an excellent review of fundamentals, and variety and stimulate the curiosity of the pupils about the very skills they are practicing. Most of these exercises will interest the majority of students, but
they cover many fields so as to interest every student of solid geometry. It is not intended that every pupil work every problem. Exercises are classified according to subject or topic in order to make it easy for the student and teacher to find quickly any kind of exercise desired. These may be used in classroom or for daily preparation to care for individual differences and interests of students. If these problems are used for review or tests, the teacher should draw from the various lists to encourage the use of initiative and resourcefulness on the part of the pupil. At all times the practical applications of solid geometry have been the guiding principle.

If this manual were to be published, more review exercises could be added at the end of each unit. These exercises are not functional problems, but they will be useful in teaching solid geometry. True-false, multiple-choice, best answer, and completion tests give a diversity of exercises for needed reviews. This part of the manual could be expanded. This study might encourage some teacher of trigonometry to make a similar manual.

Functional exercises, as have been worked out for this study, show one type of material which would vitalize any solid geometry course. Such a manual gives the best opportunity for successful work at the hands of a
good instructor. No manual can take the place of an enthusiastic, resourceful teacher. It merely supplements her skill. David Eugene Smith writes:

In the hands of a dull, mechanical, grad-grind person with a teacher's license, no book can be successful. The teacher who does not anticipate difficulties which would otherwise be discouraging to the pupil, tempering these difficulties (but not wholly removing them) by skillful questions, is not doing the best work. On the other hand, the teacher who overdevelops, who seeks to eliminate all difficulties, who does all of the thinking for the class, is equally at fault. Youth takes little interest in that which offers no opportunity for struggle, whether it be on the playground, in the home games of an evening, or in the class room.1

---

HISTORY AND USE OF INANCING SOLDIERS' PAYMENT

PART II
This manual contains many problems to be used as a supplement for solid geometry textbooks. The problems are classified according to geometric topics with exercises in each group pertaining to the various fields. Not every student is expected to solve each problem. There is an abundance of practice material to provide for the varying abilities and interests of individual students, as well as the variance in classes. These problems can be used in the classroom, for outside preparation, or for special assignment. If the teacher wishes, mimeographed copies can be made of the needed parts or problems. Any combination of exercises can be made by selecting problems from the desired topics. These problems can be used effectively in supervised study to give the weak student necessary review of various fundamentals without repetition of the textbook problems. Rarely does the entire class have the same difficulty, which fact makes this manual the more useful, as it can be used in many ways.

The problems of this manual have accumulated over a twelve year period of teaching solid geometry in providing for individual differences and interests of students electing this subject. It is hoped that some of the ideas presented in this manual may aid in vitalising present courses of solid geometry. The author's desire for functional problems in booklet form to supplement the textbook prompted her to compile these problems. As relatively few students take solid geometry and as the desirable combinations may be made available by mimeographed copies, the author has not intended publication. If this manual were to be published, however, more review exercises as found at the end of each unit, such as true-false, multiple-choice, best-answer, and completion exercises, could be added. These are not functional problems but are new-type tests to review the essentials in another way.

G. S. N.

Anderson, 1939.
# TABLE OF CONTENTS FOR MANUAL

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>PREFACE</td>
<td>1</td>
</tr>
<tr>
<td>TABLE OF CONTENTS FOR MANUAL</td>
<td>11</td>
</tr>
<tr>
<td>PART B. PROBLEMS USED IN TEACHING SOLID GEOMETRY</td>
<td></td>
</tr>
</tbody>
</table>

## Unit

### I. LINES AND PLANES IN SPACE

A. Problems determining lines of intersection
B. Lines perpendicular to planes
C. Review exercises
   - Planes and lines
   - Completion
   - True-false
   - Best-answer
   - Parallel lines and planes
   - Dihedral angles

### II. POLYHEDRONS, PRISMS, AND CYLINDERS

A. Problems concerning lines of prisms
B. Areas of prisms
C. Volumes of parallelopipeds
D. Volumes of prisms
E. Areas of cylinders
F. Volumes of cylinders
G. Problems solving for lengths of cylinders
H. Prisms and cylinders
I. Review exercises
   - Parallelopipeds
   - Prisms
   - True-false
   - Cylinders
   - Inscribed and circumscribed figures
   - Equivalent prisms

### III. PYRAMIDS AND CONES

A. Areas of pyramids
B. Volumes of pyramids
C. Areas of cones
D. Volumes of cones
E. Review exercises
   - Matching
   - Areas of pyramids
TABLE OF CONTENTS FOR MANUAL (CONTINUED)

Volumes of pyramids ............................................. 52
Frustums of pyramids ........................................... 53
Cones .............................................................. 54
True-False ......................................................... 56
Pyramids: Plane parallel to base .............................. 56

IV. SPHERES ......................................................... 60-75
A. Problems concerning distances on spheres ................. 60
B. Areas of spheres .............................................. 62
C. Volumes of spheres ........................................... 63
D. Problems concerning spheres .................................. 65
E. Review exercises ................................................ 67-75

Spherical polygons and lunes ..................................... 67
Completion .......................................................... 69
True-False .......................................................... 71
Best-answer ......................................................... 73
Polar triangles ...................................................... 75

ANNOTATED BIBLIOGRAPHY ........................................... 76-80

 straight shall lines on the roof so that the solution may be almost in straight lines. To do this a small, square with sides, to both sides of the two sides and straighten straight. Keep growing the ends at smaller piers, only pull it over a little from the roof and let it go, or its running back against the roof a straight line is caused by too much wind. Explain why the lines are straight.

1. Do any line made for example of a mason's straight or a carpenter's straight?

2. Why is the arrow formed by holding a piece of paper always a straight line?
4. Three cubes are respectively 12 and 19 ft. high. A diagonal space of 60 long emmers from one top corner to the other shall. Find the sequence of these between cubes, if we allow for the thickness of the structure.

UNIT 1

LINES AND PLANES IN SPACE. If the cube is

A. PROBLEMS DETERMINING PLANES

1. In shingling a roof, carpenters often mark off straight chalk lines on the roof so that the shingles may be placed in straight rows. To do this a cord, covered with chalk, is held firmly at the two ends and stretched straight. Then grasping the cord at another point, they pull it away a little from the roof and let it go. As it springs back against the roof a straight line is marked by the chalked cord. Explain why the line is straight.

2. Why are tripods used for support of a surveyor's transit or a camera?

3. Why is the crease formed by folding a piece of paper always a straight line?

4. Under what conditions will three equal wires, fastened at the same height on a smokestack, hold it rigid?
5. Three shelves are respectively 12 and 10 in. apart. A diagonal brace 44 in. long extends from the top shelf to the bottom shelf. Find the segments of the brace between shelves, if no allowance is made for the thickness of the shelves.

6. If a stick is held obliquely in a water tank and is wet to a depth of 10 in., while 24 in. of the stick are in the tank, what is the depth of the water if the tank is 20 in. deep?
B. LINES PERPENDICULAR TO PLANE

1. A ship may be steered past a region of danger by observing the angle of elevation at the ship subtended by a landmark (lighthouse) within the region. The captain has a map which contains a circle with the foot of the lighthouse as center and large enough to enclose the danger region. The size of the angle which the landmark subtends with the circle is noted on this map. The course of the ship is so directed that the angle of elevation as observed from the ship to the lighthouse is never allowed to become equal or greater than the given angle on the map. Upon what geometric truth is this based?

2. A pole is perpendicular to the ground. Guy wires from the top of the pole are fastened to the ground 10 ft. from the foot of the pole. What is true about the length of the guy wires?

3. The cables that hold a derrick rigid are fastened at the top 140 ft. from the ground. They are 220 ft. long. What distance from the foot of the derrick are they fastened to the ground?

4. What is the locus of points which are equidistant from the front and a side wall of your classroom and also
equidistant from the ceiling and floor?

5. A flag pole is 20 ft. high. A guy wire 25 ft. long is stretched from its top to a point on the ground. Assuming the ground level and the pole vertical, what is the locus of the end of the wire which is on the ground?

6. From the top of a telephone pole, 24 ft. in height, a wire 30 ft. long reaches two points on the ground each 16 ft. from the foot of the pole. Assuming the ground to be level, is the pole perpendicular to the ground?

7. What must be the relation of the face and hands of a clock to the shaft on which the hands are fixed, so that the hands may always be parallel to the face?

8. How could you use a carpenter's square to determine whether or not a post is perpendicular to a level floor?
C. REVIEW EXERCISES

EXERCISE I. PLANES AND LINES

1. Through a point on a plane, how many straight lines can be drawn in the plane?

2. Do any two straight lines necessarily lie on the same plane?

3. What can you say of a straight line that has two points in common with a plane?

4. How many planes can pass through a line?

5. How many planes could be determined by three concurrent straight lines?

6. If two parallel lines are cut by a transversal are the three lines coplanar? Explain.

7. Are the sides of a parallelogram coplanar?

8. Do three points always determine a plane? Illustrate.

9. How can you tell whether a line lies on a plane?

10. Do two non-intersecting lines ever determine a plane?

11. What is the least number of planes that can inclose space?

12. Can three planes intersect in a line?

13. How many planes can be determined by three parallel lines that are not all in the same plane?

14. How many dimensions has a point; a line; a plane; a solid?

15. Find in your classroom three lines of which each is perpendicular to each of the other two.
16. In space can there be more than one line perpendicular to a given line at a given point? Explain.

17. If each of three lines is perpendicular to the other two can a fourth line be drawn perpendicular to each of the first three?

18. If a line is perpendicular to a line in a plane, is it perpendicular to the plane?

19. Are all the lines perpendicular to a vertical blackboard horizontal?

20. Are all the planes perpendicular to a vertical plane horizontal?

21. If two straight lines in the same plane are each perpendicular to a given line, what is true about the two lines?

22. Can two lines be perpendicular to a plane from a point in the plane? Explain.

23. If a quadrilateral is skew, do the lines joining the midpoints of the sides in order form a parallelogram?

24. Find two lines in your classroom that are not parallel and do not meet.

25. Where are all the points in your classroom that are equidistant from the floor and ceiling?

26. If a line is parallel to the intersection of two planes is it parallel to each of the planes?

27. What is the locus of points in a plane at a given distance from a given point without the plane?

28. If two planes are parallel, is every line in one of the planes parallel to the other plane?

29. What is the locus of points equidistant from all points on the circle?

30. Can two skew lines both be parallel to a plane?

31. If three parallel planes intercept equal segments on one transversal, do they intercept equal segments on all transversals?
EXERCISE II. COMPLETION

In the following a straight line indicates that a word is to be supplied to complete the meaning of the sentence. A row of periods indicates more than two words are to be supplied.

1. Two opposite or vertical dihedral angles are _______.

2. Two dihedral angles are said to be _______ if they have a common edge and a common face between them.

3. If a plane is perpendicular to the edge of a dihedral angle, its intersections with the faces of the dihedral angle form ___________ of the dihedral angle.

4. Two dihedral angles are said to be complementary if their ___________ ________ are complementary.

5. If two planes are perpendicular to each other, a line drawn in one of them _________ to their intersection is perpendicular to the other plane.

6. The measure of a dihedral angle may be defined as the _____________.

7. All the points equidistant from the faces of a dihedral angle lie in a ___________ _________ the dihedral angle.

8. If each of two intersecting planes is perpendicular to a third plane, their intersection is _________ to that plane.

9. All the perpendiculars to a given straight line at a given point on the line lie in a plane that is _________

10. If two angles lying in different planes have their sides respectively parallel and lie on the same side of the line joining their vertices, the angles are _________

11. Through a given point in a given plane, one line and only one can be drawn _________ to the plane.
13. Every point equidistant from two points lies in a plane which is the _________ of the line joining the two points.

14. If a line segment is parallel to a plane, it is _________ and _________ to its projection on the plane.

15. If two parallel lines are oblique to a plane, they make _________ angles with the plane.

16. If a straight line is perpendicular to one of two intersecting planes, its projection (extended if necessary) on the other plane is _________ to the intersection of the two planes.

17. Two straight lines in the same plane are either _________ or _________.

18. A plane is _________ in extent.

19. A plane is determined if it contains a given _________ and a given point outside the _________.

20. One and only one plane is formed by _________ given intersecting straight lines.

21. A plane is determined by two _________ lines.

22. Two intersecting planes have _________ common points.

23. If a straight line lies in a plane, the plane is said to _________.

24. If a plane intersects two parallel planes, the lines of intersection are _________.

25. If a plane contains one and only one of two parallel straight lines, it is _________ to the other.

26. If a straight line is parallel to a plane, it is parallel to the _________ of the given plane with any plane that contains the line and intersects the given plane.

27. If each of two intersecting lines is parallel respectively to two other intersecting lines, the plane of the
first two is \underline{perpendicular} to the plane of the other two.

26. If a line is perpendicular to each of two lines at their point of intersection, it is \underline{coplanar} to the planes of these lines.

27. Two planes perpendicular to the same straight line
are \underline{coplanar}.

28. Two planes parallel to a third plane are \underline{perpendicular} to each other.

29. The perpendicular from a point outside a plane to a plane is \underline{perpendicular}.

30. Two lines perpendicular to the same plane are \underline{parallel}.

31. All the lines perpendicular to a vertical line are \underline{parallel}.

32. All the planes perpendicular to a horizontal plane are \underline{parallel}.

33. Two lines that are not parallel and not in the same plane are \underline{perpendicular}.

A line is \underline{perpendicular} to a plane if it is perpendicular to each line in the plane through the point.

A line is \underline{parallel} to a plane if it is perpendicular to each line in the plane through the point.

A line that has three lines \underline{perpendicular} to it is \underline{parallel} to the plane of the three lines.

A line \underline{not parallel} to any of the three lines, \underline{perpendicular} to one line at a given point, is \underline{parallel} to the plane of the three lines.

A line \underline{not parallel} to any of the three lines, \underline{perpendicular} to two lines at a given point, is \underline{perpendicular} to the plane of the three lines.
EXERCISE III. TRUE-FALSE

In the following exercises if the statement is correct write a plus sign (+) in front of exercise. If the statement is not correct write a minus sign (-) in front of exercise.

1. Parallel planes have but one point in common.

2. When a sheet of paper is folded a straight edge is always made.

3. If each of three lines is perpendicular to both of the other two, they must all pass through a common point.

4. The spoke of a wheel is perpendicular to the axle. The plane formed by the turning spoke is perpendicular to the axle.

5. Two planes coincide when they have a straight line in common.

6. All the lines perpendicular to a horizontal line will be vertical.

7. A perpendicular is the shortest distance from an external point to a plane.

8. A line which is equal to its projection on a plane is perpendicular to the plane.

9. A line is perpendicular to a plane if it is perpendicular to each line in the plane through its foot.

10. The milk stool has three legs instead of four because the three legs could be more easily placed firmly on rough ground.

11. A line can be perpendicular to each of two intersecting planes.

12. In space there cannot be more than one line perpendicular to a given line at a given point.

13. A carpenter can determine by two applications of his
square whether a piece of stud is perpendicular to the floor, parallel to a given plane.

14. A plane is determined by a straight line and an outside point.

15. A line has two dimensions.

16. No more than two planes can pass through the same straight line.

17. Two skew lines can be drawn perpendicular to each other.

18. An oblique line and its projections on a plane determine a plane perpendicular to the given plane.

19. A trihedral angle can have face angles of $100^\circ$, $60^\circ$, and $40^\circ$ respectively.

20. If the projections of two lines on a plane are parallel the lines are parallel.

21. A plane perpendicular to the edge of a dihedral angle is perpendicular to each face.

22. Two planes parallel to the same line are parallel.

23. Between any two skew lines there is one and only one common perpendicular.

24. If the projections of two line segments upon a plane are equal the two lines are equal.

25. A line can be perpendicular to each of two intersecting lines.

26. Two lines perpendicular to the same plane are parallel.

27. A line perpendicular to a line of a plane is perpendicular to the plane.

28. Two skew lines can be such that one is vertical and the other horizontal.

29. There can be two planes perpendicular to the same plane through a given point in the plane.
30. One and only one line can be drawn through a given point parallel to a given plane.

31. If two angles have their sides parallel they are equal.

32. Two skew lines can both be vertical.

33. Two planes perpendicular to the same line are parallel.

34. Two planes coincide if they have a straight line in common.

35. The diagonals of a skew quadrilateral intersect.

36. Perpendiculars to two intersecting planes from a point not in either of them lie in the same plane.

37. Two lines each perpendicular to the same line, are parallel.

38. If two lines from the same point to a plane have equal projections upon the plane, they are equal.
EXERCISE IV.  BEST-ANSWER

In each exercise underline the expression that completes the sentence correctly.

1. All plane angles of a dihedral angle are
   a. obtuse  c. equal
   b. unequal  d. right angles

2. If the plane angle of a dihedral angle is a right angle, its faces are
   a. parallel to each other
   b. intersect twice
   c. equal
   d. perpendicular to each other

3. Through a straight line oblique to a plane there can be passed, perpendicular to the plane
   a. any number of planes
   b. only three planes
   c. only two planes
   d. only one plane

4. Two dihedral angles are necessarily congruent if their plane angles are
   a. acute  c. unequal
   b. equal  d. obtuse

5. The sum of two face angles of a trihedral angle is
   a. equal to the third
   b. less than the third
   c. more than four right angles
   d. greater than the third

6. If the face angles and the dihedral angles of one trihedral angle are equal respectively to the face angles and the dihedral angles of another and arranged in the reverse order, the trihedral angles are
   a. congruent  c. obtuse
   b. symmetric  d. acute
7. If two planes are perpendicular a line drawn perpendicular to one at any point in the line of intersection lies in both planes.

a. in second plane
b. in first plane
c. parallel to the second
d. partly in each

8. If two trihedral angles have the face angles of one equal respectively to the face angles of the other, the corresponding dihedral angles are

a. complementary   c. supplementary
b. unequal         d. equal

9. The sum of the face angles of any polyhedral angle is

a. equal four right angles
b. more than four right angles
c. less than four right angles
d. always constant

10. The least number of faces a polyhedron may have is

a. two   c. four
b. three d. five

11. The number of edges a tetrahedron has is always

a. three   c. five
b. four    d. six

12. The least number of vertices a prism may have is

a. five   c. seven
b. six    d. eight
EXERCISE V. PARALLEL LINES AND PLANES

1. Two straight lines that are parallel to a third line not in their plane are parallel to each other.

Given line $a$ parallel to line $b$ and line $a$ parallel to line $c$.

To prove that line $b$ is parallel to line $c$.

Through any point $P$ on line $c$ pass a plane $MP$ through line $b$. State the reason why this is possible.

Pass a plane $XY$ through lines $a$ and $b$.

Pass a plane $XY$ through lines $a$ and $c$. State the reason why this is possible.

Let planes $XY$ and $MP$ meet in line $d$ which passes through $P$.

Why?

Line $a$ is parallel to line $c$ and line $a$ is parallel to line $b$.

Why?

Line $a$ is parallel to plane $MP$.

Why?

Line $a$ is parallel to line $d$.

Why?

Therefore line $c$ and line $d$ coincide.

Why?

Line $b$ is parallel to plane $XY$.

Why?

Line $b$ is parallel to line $d$.

Why?

Therefore line $b$ is parallel to line $c$.

Why?
EXERCISE VI. DIHEDRAL ANGLES

1. If a plane bisects a dihedral angle, every point in the plane is equidistant from the faces of the angle.

Given the plane MN bisecting the dihedral angle formed by the planes NQ and NP, SR and ST are perpendiculars drawn from any point as S, in plane MN, to the planes NP and NQ.

To prove every point in MN is equidistant from NP and NQ.

Does that mean to prove $SR = ST$?

Suppose a plane were passed through SR and ST to intersect plane NP in the line AR and the plane NQ in the line AT.

What relation would plane SRT bear to planes NP and NQ?

Why?

Is $NA$ the intersection of NP and NQ?

Would plane SRT be perpendicular to NA?

Why?

What measures the dihedral angle $M-NA-P$?

What measures the dihedral angle $M-NA-Q$?

Will angle $SAR = angle SAT$?

Why?

Why does $M-NA-P = M-NA-Q$?

What relation do the right triangles SAR and SAT bear to each other?

Why?

Why would $SR = ST$?
UNIT II

POLYHEDRONS, PRISMS, AND CYLINDERS

A. PROBLEMS CONCERNING LINES OF PRISMS

1. What is the longest bar that you can put in a box 40 ft. long, 30 ft. wide, and 15 ft. high?

2. Can an umbrella which is 32 in. long be put in a suitcase which has dimensions of 28 in., 16 in., and 6 in.?

3. A trunk has a base 30 in. by 17 in., and is 21 in. high. What is the longest measurement through the trunk?

4. A cubical tank 9 in. deep is filled with water to a depth of 7 in. A foot ruler resting on and oblique to the bottom just reaches the top edge of the tank. How much of the ruler is under water?

5. A light is 10 ft. from the wall. How far from the wall should a piece of cardboard be held parallel to the wall so as to cast a shadow that is four times the area of the cardboard?
B. AREAS OF PRISMS

1. A gallon of paint costing $2.50 will cover 300 sq. ft. What will it cost to paint a garage with a flat roof, if the garage is 30 ft. wide, 40 ft. deep, and 15 ft. high, without painting the roof and allowing 300 sq. ft. for windows?

2. How many square yards of surface must be tinted, ceiling and side walls, in a hall 50 ft. long, 23 ft. wide, and 15 ft. high, if one-tenth the surface is allowed for openings, as doors and windows?

3. Seven and one-half ordinary red bricks are allowed to a square foot of wall surface. How many bricks are required for a wall 10 ft. high, 22 ft. 6 in. long, and 2 bricks thick?

4. A porch has four columns, each having the form of a right prism with regular octagonal bases. If the side of the base of each is 6 in. and the altitude is 10 ft., find the total lateral area of the four columns.

5. A room is 16 ft., long, 12 ft. wide, and 9 ft. high. How many bushels of mortar will it take to plaster the four side walls and ceiling, two coats, if one bushel
covers 6 sq. yds. one coat. Allow 120 sq. ft. for doors and windows.

6. If sheet lead 1/16 in. thick weighs approximately 3.7 lbs. to the square foot, how many pounds will be required to line the four sides and bottom of a tank 3 ft. 6 in. deep, 4 ft. wide, and 6 ft. long, making no allowance for waste or overlapping?

7. If copper weighs 1.55 lbs. per square foot, find the weight of enough copper to line the sides and bottom of a tank 30 in. long, 12 in. wide, and 10 in. deep, adding 1/2 sq. ft. for overlapping and waste.

In 20 ft. 6 in. width, and 6 ft. 3 in. length. Find the weight:

5. How long must a weak mix be kept 8 ft. wide and 6 ft. deep to hold 30 tons, allowing 26 main feet to a ton?

5. A large tank was heated on the base of a stationary engine. It was 30 ft. long, 54 in. wide, and 36 in. deep. How many cubic feet does it hold if one cubic foot contains 521 cu. in.?
C. VOLUMES OF PARALLELEPIDES

1. Find how many board feet there are in a board 6 ft. long, 1 ft. wide, and 2 in. thick. A board foot is a piece of lumber 1 ft. long by 1 ft. wide and 1 in. or less thick.

2. A granary is 33 ft. long, 8 ft. 4 in. wide, and is filled with wheat to a depth of 5 ft. How many bushels does it contain, if 5/4 cu. ft. equals one bushel?

3. One cubic foot of iron weighs 450 lbs. An iron bar is 3½ in. thick, 2 in. wide, and 6 ft. 6 in. long. Find its weight.

4. What is the weight of a block of ice 22 in. by 16 in. by 12 in., if water weighs 62.5 lbs. per cubic foot, and ice is .92 as heavy as water?

5. How long must a coal bin be that is 8 ft. wide and 6 ft. deep to hold 20 tons, allowing 35 cubic feet to a ton?

6. A huge plank was needed as the base of a stationary engine. It was 60 ft. long, 54 in. wide, and 30 in. thick. How many cubic feet were there in it?

7. A water tank is 5 ft. square at top and bottom, and 5 ft. deep. How many gallons does it hold if one gallon contains 231 cu. in.?
8. How many cubic yards of earth is removed in excavating a basement 30 ft. by 24 ft. and 9 ft. deep?

9. A car loaded with potatoes to a depth of 6 ft. is 33 ft. long and 8 ft. 2 in. wide. How many bushels of potatoes are in the car if a bushel weighs 60 lbs., and uses 5/4 cu. ft. of space?

10. A rectangular tank is 2 ft. by 4 ft. by 6 ft. What are the dimensions of a tank that will hold eight times as much?
D. VOLUMES OF FRISHS

1. A railroad embankment is to be built between vertical walls 60 ft. apart. Its cross-section is a trapezoid whose bases are 12 ft. and 24 ft., and whose altitude is 15 ft. How many cubic yards of earth are needed?

2. The plan for a canal 500 ft. long shows a cross-section 9 ft. deep, 12 ft. wide at bottom, and the retaining walls sloping outward at an angle of $45^\circ$. If the contractor estimates 30% per cubic yard to make the excavation and adds 10% for profit, what should be his bid for the job?

3. A rectangular swimming pool is 30 ft. long and 30 ft. wide. The bottom is a plane surface sloping so that the pool is 3 ft. deep at one end and 12 ft. deep at the other. How many cubic feet of water will the pool hold?

4. A trough is made of a piece of tin 6 ft. square, by folding it across the middle to form a right angle, and soldering pieces across the ends. How many cubic feet of water will the trough hold?

5. A corn crib 80 ft. long is 8 ft. high, 12 ft. wide at bottom, 18 ft. at the top, and is piled up to a height of 4 ft. above the top of the crib, making the cross-section
of this part an isosceles triangle with altitude 4 ft. 
Allowing one bushel for each 1\(\frac{3}{4}\) cu. ft., how many bushels does it contain?

6. Water stands 30 ft. deep in an hexagonal well. 
If each side of the well is 4 ft., how many cubic feet of water are there in the well?

7. A railroad embankment 120 yds. long, has a cross-
section in the form of an isosceles trapezoid, 300 ft. 
wide at base, 200 ft. at the top, and 20 ft. high. How 
many cubic yards of earth does the embankment contain?

8. A ditch with water flowing at a speed of one foot 
per second is 3 ft. deep, 6 ft. wide at top, and 3 ft. 
wide at the bottom. Allowing 7\(\frac{1}{2}\) gallons to the cubic foot, 
find the flow of the ditch in gallons per second.

9. A swimming tank is 100 ft. long, and 50 ft. wide. 
At one end the depth of the water is 4 ft., and increases 
gradually to 10 ft. at the other end. How many cubic feet 
of water are required to fill the tank?

10. A corn crib in the shape of a regular hexagonal 
prism has a base edge of 5 ft. The crib holds 500 bu. of 
corn. How many bushels of corn will a similar crib hold 
if the edges of the base are made 6 ft. long?

11. Find the cost to dig a ditch 12 rods long, 4 ft. 
deep, 7\(\frac{1}{2}\) ft. wide at top, and 4\(\frac{1}{2}\) ft. wide at bottom, at 
40\(\frac{1}{2}\) per cubic yard.
X. AREAS OF CYLINDERS

1. How many square inches of tin are required to make a cylindrical bucket having a radius of 4 in., and a height of 14 in., if 10% for waste is allowed?

2. How many 2 in. water pipes can be supplied by a 36 in. main?

3. A concrete roller is 6 ft. long and 26 in. in diameter. What area does it cover in 400 revolutions?

4. In a steam engine there are 100 cylindrical pipes, each 1½ in. in diameter and 10 ft. long, through which water passes. Find the total heating surface of these pipes.

5. A cylindrical tank car is 32 ft. long and 5 ft. in diameter. How many square feet of sheet metal are needed to make this tank car?

6. Find total area of a hot water tank whose diameter of base is 14 in., and whose altitude is 18 in.

7. Find the number of square feet in the outside surface of a smoke stack 40 ft. high and 4 ft. in exterior diameter.

8. A room is heated by 6 steam pipes each 30 ft. long
and 3 in. in diameter. What is the area in square feet of the radiating surface?

9. A tinsmith received an order for 42 stove pipes 6 in. in diameter and 5 ft. long. How many square yards of tin did he use, allowing 10% for waste?

10. Find the area of the lateral surface of a 15 gal. cylindrical gasoline tank which is 12 in. in diameter. Allow 231 cu. in. to the gallon.

11. How many 4 in. water mains will a 40 in. trunk main supply?

12. How many square inches of tin are required for making an open cylindrical pail 10 in. in diameter and 12 in. deep?

13. Find lateral area of a circular tank if the perimeter of the right section is 42 in. and the length of an element is 12 in.

14. In a steam engine there are 84 cylindrical pipes or flues, each 2 in. in diameter and 14 ft. long. These convey the heat to a fire box through the water. How much heating surface do they have?
F. VOLUMES OF CYLINDERS

1. How many gallons of water will a cylindrical hot water tank hold, whose altitude is 4 ft. and whose diameter is 15 in.? Allow \( \frac{7}{6} \) gallons to the cubic foot.

2. What are the dimensions of a quart tomato can if its altitude is equal to the diameter of its base? Allow 231 cu. in. to the gallon.

3. Find the volume of a cylindrical silo whose altitude is 20 ft. and whose diameter is 12 ft. Allowing 48 cu. ft. to a ton, how many tons of ensilage will it hold?

4. What is the weight of an iron shaft 4 in. in diameter and 16 ft. long, if a cubic foot of iron weighs 490 lbs.?

5. If a cubic foot of marble weighs 150 lbs., find the weight of a cylindrical marble column 20 ft. high and 24 in. in diameter.

6. How many cubic feet of water can be held in \( \frac{3}{4} \) ft. of pipe whose radius is 2 inches?

7. A hollow cylinder of cast iron is 22 ft. long and 2 in. thick with outside diameter 38 in. If 1 cu. in. of
8. Find the weight of a cast iron water pipe 30 ft. long, outside diameter 2 in., if the thickness of the pipe is 1/4 in.

9. How many gallons of oil will a cylindrical oil tank hold that has a diameter of 50 ft. and is 24 ft. high? Allow 7½ gallons to a cubic foot.

10. Find the cost of digging a well 100 ft. deep and 6 ft. in diameter at an average cost of $3.00 per cubic yard.

11. How many cubic yards of dirt must be excavated in digging a well 5 ft. in diameter and 45 ft. in depth?

12. A hollow cast iron roller has an inside diameter of 14 in., the thickness of the metal is 2 in., and the length of the roller is 4 ft. Find the weight of the roller.

13. Find the capacity in gallons of a tank car 32 ft. long and 5 ft. in diameter. Allow 7½ gallons to the cubic foot.

14. A winchester bushel fills a cylindrical vessel 12½ in. in height, 10 in. in diameter. How many cubic inches are there in a winchester bushel?
1. A cylindrical gasoline tank has a diameter of 25 in., a point exactly opposite an upper rim. Find how much of
how high must it be to hold 50 gallons? Allow 7\(\frac{1}{2}\) gallons
per cubic foot.

2. What is the height of a tank which contains 20 bbls. of fuel oil and has a diameter of 6 ft? Allow 31\(\frac{1}{2}\) gallons
to a barrel and 7\(\frac{1}{2}\) gallons to the cubic foot.

3. How many feet of wire 1/10 in. in diameter can be
made from 200 cu. in. of material?

4. A cylindrical can has a diameter of 6 in. and a
capacity of 4 gallons. What is its altitude?

5. A cylindrical tank is to be made 25 in. in diameter
and high enough to hold 140 gallons. How high must it be
made? Allow 231 cu. in. to the gallon.

6. A 50 gallon tank is 50 in. in diameter. What is
its height allowing 231 cu. in. to a gallon?

7. How many feet of iron piping can be made from 4500
cu. in. of iron if the inner and outer diameters of the
pipe are 6 in. and 8 in. respectively?

8. A cylinder of revolution whose base is a radius of
6 in. is equivalent to a cube having an edge of 15 in.
What is the altitude of the cylinder?

9. The diameter of the base of a tin can is 6 in., and the height of the can is 8 in. The can is filled with water to a depth of 5 in. A pencil rests obliquely in the can touching a point on the circumference of the base and a point exactly opposite an upper rim. Find how much of the pencil is under water.

10. How high must a tin can be made to hold one quart if the diameter is 3 inches? One gallon is equal to 231 cu. in.
1. A cylinder 4 in. in diameter and 6 in. long is being turned on a lathe from a block of wood 4\(\frac{1}{2}\) in. square and 6 in. long. How much wood is wasted? 6 ft. Find how many gal.

2. To find volume of an article of irregular shape, it is immersed in water in a cylinder 10 in. in diameter. The immersion causes the water in the cylinder to rise 3 in. Find the volume of the irregular solid.

3. How many feet of copper wire 1/6 in. in diameter will a cubic foot of copper make?

4. When a casting is submerged in a right circular cylindrical vessel the water rises 6 inches. If the diameter of the vessel is 14 inches, what is the volume of the casting?

5. A grindstone is 3 ft. 2 in. in diameter, and 3 in. thick. It has a square hole in the center 3 in. on the side. At 142 lbs. per cubic foot, what is its weight?

6. An irregular piece of iron in immersed in a cylindrical jar of water whose diameter is 10 in. The depth before and after the iron was put in are 8 in. and 10\(\frac{1}{2}\) in., respectively. Find volume of the iron.

7. A log 12 ft. long and 3 ft. in diameter has a defect
on the surface which causes a waste of a slab. The defect extends the entire length of the log and covers one-fourth of the surface. Find the number of cubic feet wasted in the slab. (Area of right section is difference between area of sector of circle and area of triangle)

8. A cylindrical gasoline tank lying on its lateral edge is filled to a depth of 3 ft. If the tank is 10 ft. long and the radius of the base is 2 ft. find how many gallons of gasoline the tank contains. Allow 251 cu. in. for one gallon of gasoline.

9. Rain is allowed to fall freely into a container with vertical sides. The depth of the water is the rainfall. A flat metal roof is 25 ft. by 30 ft. and a conductor pipe leads to a cistern 10 ft. in diameter. When it ceased raining the cistern was 3 ft. deep. What was the rainfall?

10. A log 4 ft. in diameter and 16 ft. long has a defect due to decay which causes a waste of a part of the log whose right section is a section of a circle of which the angle at center is 60°. Find number of cubic feet wasted.

11. A railroad tunnel has for its cross-section a rectangle 16 ft. by 10 ft. surmounted by a semi-circle 16 ft. in diameter. The tunnel is 520 ft. long. How many cubic feet of earth were removed?
E X E R C I S E  I. PARALLELEPIPEDS

1. The total surface of a cube is 486 sq. in. Find its diagonal.

2. Find ratio of the volumes of two rectangular parallelepipeds both of which are 12 in. high and the bases are 5 in., by 6 in., and 9 in. by 10 in., respectively.

3. The diagonal of a cube is \(3\sqrt{3}\). Find the area of the cube.

4. Find the length of the diagonal of a cube whose edge is 10 inches.

5. How does the lateral area of a right prism change if its altitude and the perimeter of its bases are both doubled?

6. Prove that the diagonals of a right rectangular parallelepiped are equal.

7. How does the total area of a right rectangular solid change if each of the edges meeting at one vertex is doubled?

8. The edges of a right rectangular parallelepiped meeting at a vertex are 3 in., 10 in., 12 in. Find a diagonal of the solid.

9. Find the diagonal of a right rectangular parallelepiped whose edges are 12 in., 16 in., and 18 in.

10. The total area of a right rectangular parallelepiped is 208 sq. in. If one side of the base is 6 in., and the altitude is 6 in., find the volume.

11. The volume of a right rectangular parallelepiped is 720 cu. in. The edges have the ratio of 3, 5, and 6. Find each edge.

12. The volume of a right rectangular parallelepiped is 210 cu. in. Two of its edges are 5 in., and 7 in. Find the total area.
EXERCISE II. PRISMS

1. Find lateral area of a right triangular prism, each side of base is 4 inches, and the altitude is 12 inches.

2. Find lateral area of a prism whose altitude is 10 in., and perimeter of a right section is 30 in., if the lateral edge makes an angle of 60° with the altitude.

3. Find total area of a right triangular prism whose altitude is 40 in., and each side of base is 20 in.

4. The altitude of a prism is $6\sqrt{2}$. Find the perimeter of a right section of the prism if its lateral area is 720 sq. in., and if the altitude makes an angle of 45° with the lateral edge.

5. Find the total area of a right regular hexagonal prism which has an altitude of 12 in., and each side of base is 10 in.

6. Find the lateral area of a prism whose altitude is 20 in., and the perimeter of whose right section is $24\sqrt{3}$ if the lateral edge makes an angle of 30° with the altitude.

7. The right section of a prism is a regular hexagon, whose side is 10 in. The lateral edge is 12 in. What is the lateral area?

8. Two prisms have bases whose ratio is 2 to 5 and altitudes whose ratio is 3 to 2 respectively. What is the ratio of the volumes of the two prisms?

9. Find the volume of a triangular prism whose altitude is 20 in., and the edges of the base are 7 in., 9 in., and 10 in.

10. Find the lateral area of an oblique prism whose altitude is 10 in., and the perimeter of whose right section is $10\sqrt{3}$ in., if the lateral edge makes an angle of 30° with the altitude.

11. The right section of a prism is a rhombus with diagonals 16 in. and 12 in. If the lateral edge makes an angle of 45° with the base and the altitude is 10 in., find the volume.
EXERCISE XIII. TRUE-FALSE

In the following exercises, if the statement is correct write a plus sign (+) in front of the exercise. If the statement is not correct write a minus sign (−) in front of the exercise.

1. Any section of a prism made by a plane parallel to a lateral edge is a parallelogram.

2. If two intersecting planes each contain one and only one lateral edge of a prism their intersection is parallel to the other lateral edges.

3. The lateral edges of an oblique prism are shorter than the altitude of the prism.

4. All sections of a prism made by planes parallel to the base are congruent to the base.

5. The lateral faces of a prism are parallelograms.

6. The lateral edges of a prism are perpendicular to the base.

7. The bases of a prism are always parallelograms.

8. A face angle of a trihedral angle may be greater than a right angle.

9. All right sections of any prism are congruent to the base.

10. The right section of an oblique square prism is always a rectangle.

11. All polyhedrons are prisms.

12. The lateral area of any prism is equal to the perimeter of the base multiplied by the lateral edge.

13. Two truncated prisms are congruent if the three faces which include a trihedral angle of one are congruent respectively to the three faces which include a trihedral
angle of the other, and are similarly placed.

14. If two prisms have equal altitudes and congruent bases, their lateral areas are equal.

15. An oblique prism is equal to a right prism if their right sections are congruent and their lateral edges are equal.

16. The lateral edges of a prism all have the same inclination to the plane of the base.

17. The volume of any prism is equal to the area of the base times its lateral edges.

18. The diagonals of any prism are equal.
EXERCISE IV. CYLINDERS

1. Compare the volume of two cylinders of revolution formed by revolving a rectangle whose dimensions are 12 in. by 16 in. about two adjacent sides.

2. Find the ratio of the total areas of two similar circular cylinders if the radius of one is 4 times the radius of the other.

3. A cylinder of revolution 30 in. in diameter must be how long to hold 30 gallons if 231 cu. in. equals one gallon?

4. Find the lateral area of a cylinder whose right section is a circle with 8 in. radius, and whose element is 12 in.

5. A circular cylinder has a radius of 7 in. and an altitude of 12 in. Find its volume.

6. Find lateral area of a circular cylinder whose element makes an angle of 30° with base, if the altitude is 16 in. and the perimeter of the right section is 10 in.

7. The altitudes of two similar cylinders are 4 in. and 6 in. respectively; and the lateral area of the smaller is 28 sq. in. Find the lateral area of the larger one.

8. The area of a right section of an oblique cylinder is 140 sq. in. Find its volume if its element makes an angle of 60° with its altitude of 9 in.

9. If the altitude of a right circular cylinder is constant, how does the volume change when the radius is doubled?

10. The element of an oblique circular cylinder makes an angle of 60° with the altitude. If the radius of the circle is 5 in. and the element is 20 in., find the volume of the cylinder.

11. If the dimensions of one cylinder of revolution are four times those of a similar cylinder, what is the ratio of their volumes? What is the ratio of their lateral areas?
EXERCISE V. INSCRIBED AND CIRCUMSCRIBED FIGURES

1. Find lateral area of circular cylinder inscribed in a cube whose edge is 7 inches.

2. Find the volume of a right circular cylinder inscribed in a right regular hexagonal prism each side of whose base is 20 in., and the altitude of prism is 14 in.

3. Compare the volume of a right square prism inscribed in a cylinder of revolution to the volume of a cylinder of revolution if the diameter of its base is 8 in.

4. Compare the volume of a regular triangular prism inscribed in a cylinder of revolution to the cylinder of revolution if its radius of the base is 5 in.

5. Find the lateral area of regular triangular prism circumscribed about a cylinder of revolution whose altitude is 14 in., and the diameter of the base is 20 in.

6. Find the volume of a circular cylinder inscribed in a right regular triangular prism whose base is 20 in., on a side.

7. What is the ratio of volume of a cylinder to the volume of the regular quadrangular prism circumscribed about it?

8. What is the ratio of volume of a regular triangular prism to the volume of the circumscribed cylinder?
EXERCISE VI. EQUIVALENT PRISMS

1. An oblique prism is equivalent to the right prism whose base is a right section of the oblique prism and whose altitude is equal to a lateral edge of the oblique prism.

Given FK, a right section of the oblique prism AS; also FY, a right prism whose lateral edges equal the lateral edges of AS.

To prove oblique prism AS equivalent to right prism FY.

From the lateral edges of AS and FK subtract the lateral edges of FY. Will AF = OV?

What will BQ equal? ...

What will CH equal? ...

From the definition of a prism will the upper bases FK and FY be equal? ...

If truncated prism AK be placed on truncated prism OV so that FK coincides with VY will FA coincide with VQ? ...

CB with FV? ...

Will these coincide because only one perpendicular can be drawn through a given point to a given plane? ...

Why will they coincide? ...

Do faces AG and OV coincide? ...

Why? ...

Does face BK coincide with face FK? ...

Why? ...
Do bases \( FK \) and \( VY \) coincide? Why? Why? \\
When are two truncated prisms congruent? \\
For this reason truncated prisms \( AK \) and \( OY \) are congruent. \\
But \( AK + FS = AS \). Why? \\
And \( OY + FS = FY \). Why? \\
Since the sums are equal when equals are added to \( AS \) is equivalent to FY. \\
2. When are two solids equivalent? \\
3. Are all equivalent solids congruent? \\
4. Are all congruent solids equivalent? \\
5. Compare the lateral edge and the altitude of a right prism. \\
6. What kind of figures are the lateral faces of any prism? of a right prism?
UNIT III

A. AREAS OF PYRAMIDS

1. A silk lamp shade is made in the shape of the frustum of a right regular hexagonal pyramid whose bases are 4 in. and 10 in. respectively on each face. If the lateral edge is 10 in., how much silk is necessary to make two shades?

2. The slope of the roof of an hexagonal tower is 45°. If each side of the base is 8 ft., find the length of the rafters needed which are lateral edges and slant heights of each face. Find surface of the roof of the tower.

3. A coal chute running from a window to the hopper of a coal stoker is in shape of a frustum of a right regular square pyramid. The base at window is 16 in. on a side and the other base is 25 in. on a side. The lateral edge is 4 ft. How much material is needed to make the chute?
4. A card measuring 3 in. by 5 in. is held parallel to the wall. If a flash-light is held directly back of the

card 6 in. from the card and 16 in. from the wall, find the area of the shadow cast by the card on the wall.

5. Two tents are made in the shape of regular square pyramids, one 12 ft. on a side and 16 ft. high, and the other 10 ft. on a side and 12 ft. high. Find amount of canvas used for both.

6. A square roof is in the form of a frustum of a pyramid, the upper base being a flat deck which is 6 ft. on a side and lower base 18 ft. on a side. The height of roof is 8 ft. What is the area of roof?

7. Find the area of a shadow that a book 8 in. wide and 10 in. long will cast if the book is held parallel to the wall. The light is 5 ft. from the book and 13 ft. from the wall.
B. VOLUMES OF PYRAMIDS

1. A wastepaper can is in the form of a parallelepiped surmounted by regular square pyramid. The total height of the can is 4 ft., the height of the rectangular part is 3 ft., and each side of the square base is 10 in. Find total volume.

2. A coal bin is in the form of a rectangular solid with one side square base 14 ft. and height 10 ft. Below this is a frustum of a regular square pyramid whose lateral edges slope 45°. The lower base is 2 ft. square. Allowing 40 cu. ft. for one ton, how many tons does this bin hold?

3. A strawberry box is 5 in. square at the top and 4 in. square at the bottom. If the box is 3 in. deep and one quart is equivalent to 67 cu. in., does the box hold a quart?

4. An obelisk in the form of a quadrangular frustum is 64 ft. high; 8 ft. square at the base, 5 ft. square at the top, and surmounted by a pyramid 7 ft. high. A cubic foot of stone weighs 180 lbs. How many tons does this figure weigh?

5. A wooden tank is 6 ft. square at the bottom, 5 ft. square at the top and 4 ft. deep. How many gallons of
water will it hold if 231 cu. in. is allowed for each gallon.

6. A monument is in the form of a frustum of a regular quadrangular pyramid 10 ft. in height, the side of whose bases are 8 ft. and 6 ft. respectively, surmounted by a regular pyramid 6 ft. in height, each side of whose base is 6 ft. Find weight of monument, if it weighs 150 lbs. to the cubic foot.

7. Two tents are made in the shape of regular square pyramids, one is 12 ft. on a side and 8 ft. high, and the other is 16 ft. on a side and 10 ft. high. Find the ratio of their air capacities.

8. A farmer has two similar rectangular ricks of hay in shape of a pyramid. The large rick was 1 1/2 times as long as the small one. He hauled away the small one and found it contained 2 1/2 tons. How many tons did the large rick contain?

9. A farmer has 2 similar wedge-shaped piles of corn. One was 10 ft. high and the other only 4 ft. high. The small pile was known to contain 70 bu. How many bushels in the large pile?

10. A marble column is in the form of a frustum of a right square pyramid. The column is 12 ft. high and the sides of the bases are 3 ft. and 1 1/2 ft. respectively. If marble weighs 150 lbs. to one cubic foot, how much does the column weigh?
6. Areas of Cones

1. How much canvas is required for a conical tent 8 ft. in height if the diameter of the base is 10 ft. and 10% is added for seams and overlapping?

2. A cone-shaped hood for a chimney is to be 36 in. in diameter and 14 ft. for slant height. What surface of metal is required to make the hood adding 10% for overlapping and waste?

3. A tin basin is 14 in. in diameter at the top and 12 in. in diameter at the bottom. If the slant height is 4 in., how many square inches of tin are required to make it?

4. A flagpole 60 ft. high stands on level ground. What is the area of the circle on the ground made by locus of points of a rope 100 ft. long attached to the top of the pole?

5. Find the number of cubic feet in a log 33 ft. long and 36 in. in diameter at the small end and 48 in. at the large end.

6. A funnel is made of tin, in which the diameter at the top is 6 in. and at small opening 1 in. and at extreme small and 1/2 in. If the length (slant height) of each part is 5 in., how much tin is required to make it if 10%
is added for seams?

7. A copper teapot is 6 in. in diameter at the bottom, 4 in. in diameter at the top and 6 in. deep. Adding 10% for overlapping and waste, how much metal is required for its construction without the cover?

8. How many square feet of tin are needed to make a funnel with top diameter 28 in., bottom diameter 4 in., and whose slant height is 22 in.?
the sides of a bin. The highest point of the wall is 3 ft. above the floor. The radius of the bin and the slant height of the bin are 8 ft. Find the capacity of each of the bins.

D. VOLUMES OF CONES

1. A farmer has a conical shaped heap of grain on his granary floor. The height of the heap is 5 ft., and the circumference of the circle covered by its base is 30 ft. Allowing 1/4 cu. ft. to bushel, how many bushels of grain are in the heap?

2. Find the weight of a solid circular cone of cast iron whose height is 10 in. and the diameter of whose base is 8 in., if 1 cu. in. of cast iron weighs .30 lbs.

3. How many ice cream cones 2 in. in diameter and 3 in. high can be filled from a gallon of ice cream? Allow 231 cu. in. to a gallon.

4. How many bushels of wheat are there in a heap thrown into the corner of a rectangular bin if the highest point is 3 ft. above the floor and the distance of any point on the edge of the heap from the corner on the floor is 6 ft.?

5. A piece of tin in the form of a sector of a circle of radius 10 in. is rolled into a cone. If the central angle of the sector is 180°, what is the volume of the resulting cone?

6. A load of small pieces of coal is thrown up against
the side of a bin. The highest point of the coal is 4 ft. where it touches the sides of the bin and the slant height of the heap is 8 ft. If the base of the heap is semi-circular in form, how many tons does it contain allowing 36 cu. ft. per ton?

7. A tank in the form of a frustum of a right circular cone is 14 ft. in diameter at the bottom, 10 ft. in diameter at the top and 9 ft. deep. How many cubic feet of water will it hold?

8. A smokestack made of brick is 130 ft. high, 20 ft. in diameter at the ground, and 12 ft. at the top. It has a cylindrical flue 4 ft. in diameter throughout its length. Allowing 22 bricks per cu. ft., how many bricks are there in it?

9. A water tank is 6 ft. in diameter at the top, 7 ft. in diameter at the bottom, and 4 ft. 6 in. deep. How many gallons of water will the tank hold?

10. A right circular cone of wood has a cylindrical hole 10 in. in diameter bored entirely through it, the axis of the cylinder coinciding with the axis of the cone. The altitude of the cone is 24 in. and the altitude of the cylinder is 12 in. Find the amount of wood left.
EXERCISE I. MATCHING

Below is a list of terms and their definitions. Match them by placing the letter from the right column in front of its corresponding term in the left column.

1. Altitude of a frustum of a pyramid.
   a. The planes forming the boundary surfaces of a polyhedron.

2. Pyramid.
   b. A pyramid whose base is a regular polygon, and whose altitude passes through the center of the base.

3. Altitude of a pyramid.
   c. The altitude of any one of the lateral faces of a regular pyramid.

   d. The perpendicular distance between the bases of a frustum of a pyramid.

5. Faces of a polyhedron.
   e. A polyhedron formed by joining each of the vertices of a polygon to a point not in the plane of the base.

6. Lateral area of a frustum of a right regular pyramid.
   f. The sum of the areas of the lateral faces of a pyramid.

7. Slant height of a right regular pyramid.
   g. The perpendicular distance from the vertex to the base of a pyramid.
8. Regular pyramid.

9. Lateral area of a right regular pyramid.

10. Frustum of a pyramid.

11. Lateral area of a cone of revolution.

12. Altitude of a cone.

13. Lateral area of a frustum of a cone of revolution.


15. Slant height of a right circular cone.

16. Frustum of a cone.

17. Circular cone.

18. Altitude of a frustum of a cone.

h. The area of the base of a pyramid times 1/3 of its altitude.

i. The portion of a pyramid between its base and a plane parallel to its base.

j. Equals half the sum of the perimeters of the bases multiplied by the slant height of the frustum of a right regular pyramid.

k. A cone whose base is a circle.

l. Any element of a cone of revolution.

m. The product of one-half the element times the circumference of the base of a right circular cone.

n. Cones which have their corresponding segments in the same ratio.

o. Equal the circumference generated by the midpoint of an element multiplied by the slant height of the frustum of the right circular cone.

p. The lateral area of a cone of revolution added to area of the base.

q. The perpendicular distance from the vertex to the base of a cone.

r. The area of the base multiplied by 1/3 the altitude of a cone.
10° Standard cone.

Base of cone.
EXERCISE II. AREAS OF PYRAMIDS

1. Find lateral area of a right regular triangular pyramid whose slant height is 10 in., and each side of base is 8 in.

2. One side of base of a right regular quadrangular pyramid is 10 in., and the altitude of the pyramid is 5 in. Find total area.

3. Given a right regular octagonal pyramid with lateral edge equal 10 in., and side of base 12 in. Find lateral area.

4. Given a regular hexagonal pyramid with altitude 6 in. and a side of base 10 in. Find total area.

5. A cube has 12 in. edges. In each face connect the midpoints of adjacent sides. What is the total area of the polyhedron formed by cutting off the corners of the cube?

6. The altitude of a right regular triangular pyramid is 18 ft., and each side of the base is 6 ft. Find the area of a section parallel to the base and 5 ft. from the vertex.

7. The altitude of a right regular quadrangular pyramid is 7 in., and each side of the base is 8 in. Find the area of a section parallel to the base and 3 in. from the base.

8. In a right regular hexagonal pyramid, if the altitude is 8 ft., and a side of the base is 10 ft., find the area of a section parallel to the base and 2 ft. from the vertex.

9. In a regular hexagonal pyramid, if the altitude is 12 in., and a side of the base is 8 in., how far from the vertex must a plane be passed parallel to the base if the area of the section formed is $40\sqrt{3}$ sq. in.?

10. In a pyramid whose altitude is 20 in., how far from the vertex must a plane be passed parallel to the base so that the area of the section is one-half the area of the base?
EXERCISE III. VOLUMES OF PYRAMIDS

1. Given a regular hexagonal pyramid with side of base 12 in., and altitude 6 in., find volume.

2. If a regular quadrangular pyramid has 8 in., for its slant height and each side of base is 10 in., find volume.

3. If the inclination of a 12 in., lateral edge to the base of a right hexagonal pyramid is 30°, find volume.

4. The inclination of the lateral face to the base of a right regular quadrangular pyramid is 45°. If the slant height is 5 in., find volume.

5. Find volume of a pyramid if the sides of the base are 10 in., 12 in., 14 in., and its altitude is 8 in.

6. The angle of inclination of the lateral face to the base of a right regular quadrangular pyramid is 30°. If the slant height is 14 in., find volume.

7. Find the volume of a regular tetrahedron whose edge is 20 in.

8. Find the altitude of a pyramid with a square base if the volume of the pyramid is 144 cu. in., and one side of the base is 6 in.
EXERCISE IV. FRUSTUMS OF PYRAMIDS

1. Find the volume of the frustum of a right regular quadrangular pyramid if the sides of the bases are 5 in. and 9 in., respectively and the altitude is 4 in.

2. Given the frustum of a right regular triangular pyramid whose side bases are 4 in. and 16 in., respectively. If the altitude is 10 in., find the volume.

3. Given the frustum of a right regular quadrangular pyramid with sides of bases 14 in. and 22 in. If lateral edge is 20 in., find total area.

4. Find the volume of the frustum of a right regular hexagonal pyramid if the sides of the bases are 12 in. and 20 in. and the lateral edge is 16 in.

5. The areas of the upper and lower bases are 4 sq. ft. and 9 sq. ft., respectively. If the volume of the frustum is 200 sq. ft., find the altitude.

6. If the areas of the bases of a frustum of a pyramid are 9 sq. in. and 16 sq. in., respectively, and the altitude is 3 in., find the volume.

7. If a plane is passed parallel to the base and 5 in. from the base of a right regular hexagonal pyramid whose altitude is 20 in. and the edge of whose base is 12 in., find one side of the section. What is ratio of area of section to area of the base?

8. The altitude of a frustum of a right regular quadrangular pyramid is 8 in. and the sides of the bases are 10 in., and 22 in. Find lateral area and volume.

9. The side of the base and the altitude of a right regular hexagonal pyramid are 10 in. and 12 in., respectively. A plane is passed through it parallel to the base 8 in. from the vertex. Find one side of section made by plane. Compute the volume of the large pyramid.

10. The height of a frustum of a regular square pyramid is 4 in. and the sides of the bases are 10 in. and 16 in. Find volume and lateral area.
EXERCISE V. CONES

1. The slant height of a right circular cone is 13 in., and the radius of the base is 5 in. Find the lateral area and the total area.

2. Find the volume of a cone of revolution if the base of the generating right triangle is 10 in. and the altitude is 12 in.

3. An isosceles triangle with altitude 6 in. and base 10 in. revolves about its altitude. Find the volume of the cone generated.

4. Two similar cones of revolution have radii in the ratio 2:3. Find the ratio of their lateral areas.

5. The radius of the base of a cone of revolution is 15 in., and the axis of the cone is 36 in. Find the length of an element.

6. The radius of a cone of revolution is 12 in. Find the radius of a similar cone of revolution having a lateral area four times as large.

7. The radii of the bases of a frustum of a right circular cone are 6 in. and 9 in., respectively, and the altitude is 4 in. Find the lateral area.

8. Find the volume of a cone of revolution whose total area is 124 sq. in., and whose base has a radius of 3 in.

9. The height of a right circular cone is 8 in., and the area of its base is 24 sq. in. Find its volume.

10. Find the total area of a cone of revolution whose slant height is 12 in., and whose base has a radius of 4 in.

11. Find the lateral area of the frustum of a cone of revolution if the radii of the bases are 9 in. and 12 in., and the altitude is 4 in.

12. Find the volume of the frustum of a right circular
cone whose altitude is 2 in., and the radii of whose bases are 3 in. and 4 in.

13. Find the lateral area of a right circular cone if the radii of the bases are 7 in. and 8 in., and the slant height is 5 in.

14. If the altitudes of two similar right circular cones are in the ratio 1:3, find the ratio of their volumes.

15. A semi-circular piece of paper 6 in. in radius is folded into the form of a conical surface. Find the volume of the cone.

16. The lateral areas of two similar cones are 144 sq. in. and 225 sq. in., respectively. If the altitude of the first cone is 4 in., find the altitude of the second cone.
EXERCISE VI. TRUE-FALSE

10. In the following exercises if the statement is
   correct write a plus sign (+) in front of exercise. If
   the statement is not correct write a minus sign (−) in
   front of the exercise.

1. Every pyramid is a polyhedron. 

2. The lateral faces of a pyramid are triangles. 

3. The element and slant height of a cone of revolu-
   tion are equal. 

4. A section determined by two elements of a cone
   is equilateral. 

5. The base of a pyramid is a regular polygon. 

6. A triangular prism is a tetrahedron. 

7. The elements of two cones of revolution have the
   same ratio as their altitudes. 

8. The area of a section made by a plane parallel to
   the base of a cone and bisecting its altitude is one-fourth
   the area of the base. 

9. A plane parallel to the base of a pyramid makes a
   section similar to base. 

10. The areas of two similar pyramids are to each
    other as their lateral edges. 

11. The volumes of two similar cones are to each other
    as the squares of their altitudes. 

12. The lateral area of a frustum of a right circular
    cone is equal to the product of the circumference at a
    mid-section by the slant height. 

13. A pyramid may have six equilateral triangles for
    its lateral faces.
14. The bases of a frustum of a pyramid are always similar polygons.

15. The elements of a cone are equal.

16. The volume of any pyramid is one-half the product of the area of the base and its altitude.

17. The area of a section made by a plane which is parallel to the base of a pyramid and bisects its altitude, is one-half the area of the base.

18. If two pyramids with equal altitudes are cut by planes parallel to the bases, and at equal distances from their vertices, the sections have the same ratio as the bases.

19. The lateral area of a regular tetrahedron is four times the area of its base.

20. The lateral area of a cone of revolution is equal to circumference of base times one-half its altitude.

21. The vertex angle of a lateral face of a pyramid may be a right angle.

22. The altitude of a pyramid may be equal to its lateral edge.
EXERCISE VII. PYRAMIDS: PLANE PARALLEL TO BASE

1. If a pyramid is cut by a plane parallel to the base:
   (a) the edges and altitude are divided proportionally;
   (b) the section formed is a polygon similar to the base;
   (c) the area of the section is to the area of the base as the square of the distance from the vertex to the plane is to the square of the altitude of the pyramid.

Given plane $XYZ\parallel$ parallel to the base $ABCD$; plane $XYZ\parallel$ intersecting the faces in $XY$, $XZ$, $YZ$, and $NZ$; and the altitude $NO$ at $P$.

To prove:

(a) $\frac{RX}{RA} = \frac{RY}{RB} = \frac{RZ}{RC} = \frac{RN}{RD} = \frac{RP}{NO}$

(b) Polygon $XYZ\parallel$ similar $ABCD$;

(c) $\frac{\text{Area } XYZ\parallel}{\text{Area } ABCD} = \left(\frac{RP}{NO}\right)^2$

Present a plane $EF$ through $R$ parallel to plane $ABCD$. Then

(a) If two straight lines are cut by three or more parallel planes, the corresponding segments are

(b) Do the sides of the angles $MXY$ and $DAB$ lie in different planes?

Is $XY$ parallel to $AB$?

Is $XX$ parallel to $AD$?

Why?
Is the same true for angles XYZ and ABC? ...
For angles YZX and BCD? ...
For angles ZXN and CDA? ...

Then angle NXY = angle DAB etc.
Why? ...

And \[
\frac{XY}{AB} = \frac{XZ}{BC} = \frac{NX}{AD} = \frac{NY}{RB}
\]
etc.
Why? ...

Then polygon XYZN is similar to ABCD. Why? ...

(c) The areas of two similar polygons are to each other as two corresponding sides.

Then \[
\frac{\text{Area } XYZN}{\text{Area } ABCD} = \left(\frac{\text{XY}}{AB}\right)^2
\]
But \[
\frac{XY}{AB} = \frac{XZ}{BC} = \frac{NX}{AD} = \frac{NY}{RB}
\]
Then \[
\frac{\text{Area } XYZN}{\text{Area } ABCD} = \left(\frac{\text{XY}}{AB}\right)^2 = \left(\frac{\text{XZ}}{BC}\right)^2 = \left(\frac{\text{NX}}{AD}\right)^2 = \left(\frac{\text{NY}}{RB}\right)^2
\]

2. Pyramid RXYZ is similar to pyramid R-ABCD.
Why? ...

The volumes of two similar pyramids are to each other as two corresponding lateral edges.

Volume \(R-XYZ\) = \(\frac{(XY)^2}{(AB)^2}\)

Volume \(R-ABCD\) = \(\frac{(XZ)^2}{(BC)^2}\)
UNIT IV

SPHERES

A. PROBLEMS CONCERNING DISTANCES ON SPHERES

1. One aviator lives in Indiana and another aviator lives north of him in Michigan. If both fly directly west which one would fly the greater distance before returning to his home? Would their paths cross?

2. One aviator lives in Illinois and another aviator lives directly east in Indiana. If both aviators fly directly south will their paths meet?

3. One ship is in longitude 50° W, latitude 30° N, and a second ship is in longitude 50° W, latitude 30° S. Are these ships respectively north and south of each other?

4. Why do ships in going from New York to an European port due east, travel a little north of east, then finally a little south of east?

5. Considering the earth as a sphere with a radius of 3960 miles how far is it from the 56th parallel north
6. Two boards are nailed at right angles to each other forming a trough. A sphere 10 in. in diameter is placed in the trough. How far from the edge of the dihedral angle will the ball be tangent?

7. If the radius of spherical ball is 20 in., what is the circumference of a circle made by a plane 15 in. from the center of the ball?

8. How many degrees west of New York which is longitude 75° W, is the 150th meridian? How many degrees east?

9. What is the length in geographical miles of a degree of latitude on the 45th parallel?

10. What is the length in geographical miles of 60 degrees of latitude on the 30th parallel?

11. What is the length in statute miles of the circle at the 60th parallel if the radius of the earth is 4000 miles?

12. When it is noon, solar time, at a point on the 40th parallel north latitude at the time of an equinox, what is the elevation of the sun?

13. When it is noon June 21 on the 49th parallel, north latitude, what is the elevation of the sun?

14. What is the radius of a circle which is formed by a plane passed 4 in. from the center of a ball which is 16 in. in diameter?
B. AREAS OF SPHERES

1. What is the area of the earth's surface if its radius is 3960 miles?

2. Find the area of a hemispherical dome whose diameter is 20 ft.

3. What part of the surface of the earth lies in the zone between the equator and the parallel $30^\circ$ north?

4. What is the area in spherical degrees of a lune of the earth, whose angle is $72^\circ$?

5. The ratio of the radii of two spheres is 2:3. What is the ratio of their areas?

6. About 30% of the earth's surface is covered with land. Assuming the radius of the earth to be 4000 miles, find the number of square miles of land.

7. A tank consists of a cylindrical portion with hemispherical ends. The diameter of the ends equals the diameter of the cylinder, which is 2 ft. The total length of it is 7 ft. Find the total area.

8. A cube has an edge of 6 in. Find the radius of a rubber ball whose area is equal to the total area of the cube.
6. The volume of water with a hemispherical bowl whose diameter is 5 in. and whose thickness is 1 in. is allowed for a washbasin.

C. _VOLUMES OF SPHERES_

1. What is the weight of a hollow cast iron sphere whose outside diameter is 5 in. and whose inside diameter is 4 in., if 36 lbs. is allowed for a cubic inch?

2. Find the volume of a spherical shell whose outside diameter is 6 in. and whose thickness is 1 in.

3. The diameter of one orange is 25% greater than that of another. Its volume is what percent greater?

4. A steel ball bearing for a car has a diameter of 1.85 cm. If steel has a density of 7.82 grams per cubic centimeter, what is its weight in grams?

5. The amount of juice of one grapefruit is 3 times as much as that of another. What is the ratio of their diameters?

6. How many iron balls 2 in. in diameter can be made from one 15 in. in diameter?

7. A hollow spherical shell of steel is 1 in. thick and its outer diameter is 8 in. Find its weight if one cubic foot of steel weighs 490 lbs.?

8. A cylinder 8 in. in diameter is partly filled with water. When a ball is immersed in the water the surface rises 2 in. Find the diameter of the ball.
9. How many quarts of water will a hemispherical bowl hold whose diameter is 18 in., if 231 cu. in. is allowed for a gallon?

10. A rubber ball is placed inside a cube box so that the ball is tangent to the sides of the box. If one edge of the box is 231 in., find the volume of the ball.
D. PROBLEMS CONCERNING SPHERES

1. How many square yards are in the surface of a spherical balloon containing 1000 cu. ft. of gas?

2. Two spheres of lead, of radii 2 in. and 4 in. respectively, are melted and recast into a solid cylinder of revolution whose altitude is 6 in. and whose radius is 4 in. Show that the total surface is unchanged in amount.

3. A hole 1/4 in. in diameter was bored directly through the center of a lead ball 2 in. in diameter. Find the weight of the ball if a cubic foot of the lead weighs 712 lbs.

4. Find the diameter of a ball whose area is one square foot.

5. Two grapefruit of the same kind have diameters 3 in. and 4 in. respectively. What should be the selling price of the larger kind if the smaller ones sell at the rate of 3 for 25c?

6. What is the length of $10^6$ in geographical miles at the 30th parallel on the earth's surface?

7. What is the length of $5^6$ in geographical miles at the 45th parallel on the earth's surface?
8. What is the length in geographical miles of 120° at the 60th parallel on the surface of the earth?

9. A water ball has 10 in. for its radius. What is the length of the circle made by a plane passed 3 in. from the center of the ball?

10. Find the ratio of the volumes of two oranges if the ratio of their diameters is 2 in. to 3 in.

11. If the length of a side of a spherical triangle are 50°, 60°, and 70°, find the number of degrees in each side of the polar triangle.

12. If in the spherical excess of a spherical triangle whose angles contain 50°, 60°, and 70°, the number of degrees in each angle of the polar triangle is 10°, find the number of degrees in each angle of a spherical triangle whose angles contain 40°, 50°, and 60°.

13. Find the area of a spherical triangle whose angles contain 50°, 60°, and 70°.

14. The radius of a spherical triangle contains 60°, 70°, and 80° and the radius of the sphere is 6 in. Find the area of the spherical triangle expressed in square inches.

15. What is the area of a cone on a sphere having a diameter 12 in. in length of the angle of the base in 30°?

16. Find the area of a cone on a sphere having a radius of 12 in. and height of the cone is 6 in.

17. The radius of a spherical rectangle are 150°, 120°, 100°, and 80° respectively. Find the angle in each part of the sphere.
EXERCISE I. SPHERICAL POLYGONS AND LUNES

1. The angles of a spherical triangle are 90°, 75°, and 45°. Find the number of degrees in each side of its polar triangle.

2. The sides of a spherical triangle contain 80°, 90°, and 100°. Find the number of degrees in each angle of its polar triangle.

3. What is the spherical excess of a spherical triangle whose angles contain 60°, 75°, and 90°°?

4. What is the spherical excess of a polygon having angles of 110°, 70°, 90°, and 85°?

5. Find the perimeter of a trirectangular spherical triangle on a sphere having a radius of 25 in.

6. How many spherical degrees does a spherical triangle contain if its angle sum is 360°?

7. What part of sphere is a trirectangular spherical triangle?

8. How many spherical degrees are in the spherical triangle whose angles contain 70°, 90°, and 100°?

9. The angles of a spherical triangle contain 85°, 75°, and 130° and the radius of the sphere is 3 in. long. Find the area of the spherical triangle expressed in square inches.

10. What is the area of a lune on a sphere with a diameter 20 in. in length if the angle of the lune is 36°?

11. Find the area of a zone on a sphere having a radius of 12 in. if the height of the zone is 4 in.

12. The angles of a spherical polygon are 100°, 110°, 120°, and 130° respectively. The polygon is what part of the sphere?
13. The volume of one sphere is 27 times that of another. What is the ratio of their radii?

14. Find the diameter of a sphere whose area is one square foot.

15. A lune whose angle is 45° and an equilateral spherical triangle each of whose angles is 60° are on the same sphere. Find the ratio of the area of the lune to the area of the triangle.

16. Two of the angles of a spherical triangle are 50° and 90°. The angle of an equivalent lune is 45°. Find the third angle of the triangle.
EXERCISE II. COMPLETION

In the following exercises a straight line indicates that a word is to be supplied to complete the meaning of the sentence. A row of periods indicates that more than two words are to be supplied.

1. A diameter of a sphere is _________ its radius.

2. A _______ is the locus of points at a given distance from a given point.

3. Spheres with equal radii or with equal diameters are _________.

4. A quadrant of a circle contains _________ degrees.

5. If a straight line segment that has one end at the center of the sphere is shorter than the radius, it lies _________.

6. A plane that is perpendicular to a radius of a sphere at its outer extremity is _________ to the sphere.

7. The greatest number of spheres that can be passed through four noncoplanar points is _________.

8. A sphere may be inscribed in any irregular _________.

9. The area between two great circles of a sphere is called a _________.

10. The sum of the angles of a spherical triangle is greater than _________ but less than _________.

11. The sum of two sides of a spherical triangle is _________ than the _________ side.

12. The sum of the sides of a convex spherical polygon is less than _________ degrees.

13. If a plane intersects a sphere, the intersection is a _________. 
14. The formula for the area of a sphere is .......... 

15. The formula for the volume of a sphere is .......... 

16. A spherical angle is formed by two ________ circles on a sphere. 

17. If the angle of a lune is 40°, its area is ________ spherical degrees. 

18. A plane tangent to a sphere is ________ to the radius drawn to the point of contact. 

19. The formula for the area of a spherical surface is .................. 

20. The shortest distance between two given points on the surface of the sphere is .................. 

21. All great circles of a sphere are _____________ . 

22. Any two ________ circles on a sphere bisect each other. 

23. The spherical distance of all points on a circle of a sphere from a given pole of a circle are ____________ . 

24. If two symmetric spherical triangles are isosceles, they are ____________ . 

25. The formula for the area of a lune is .............. 

26. The formula for the area of a spherical triangle is .................. 

27. The center of a sphere inscribed in any tetrahedron is found by ..................
EXERCISE III. TRUE-FALSE

In the following exercises, if the statement is correct, write a plus sign (+) in front of the exercise. If the statement is not correct write a minus sign (−) in front of the exercise.

1. The equator is a great circle of the earth.

2. A plane tangent to a sphere is perpendicular to the radius drawn to the point of tangency.

3. The area of a sphere is 2 if its diameter is 2.

4. A sphere may be passed through any four points.

5. All great circles of a sphere are equal.

6. If the sum of the angles of a spherical triangle is 200°, the spherical excess is 100°.

7. There exists a triangle whose angles are 50°, 60°, 70°.

8. If an angle of a lune is 40°, the area of the lune is 40 spherical degrees.

9. The shortest distance between two points on a sphere is the arc of the great circle passing through these points.

10. Only great circles have polar distances.

11. If the angles of one spherical triangle are 70°, 80° and 90°, the sides of the polar triangle are 110°, 80°, and 90°.

12. Not more than one sphere can be passed through three points which are not in the same straight line.

13. If two lines are tangent to a sphere at a given point on the sphere, the plane of these lines is tangent to the sphere at that point.
14. The polar distance of a great circle is 90°.

15. If a sphere is inscribed in a cube, the radius of the sphere is a side of the cube.

16. The planes of two great circles of a sphere may be parallel.

17. All points of a circle on a sphere are a quadrant's distance from the poles of the circle.

18. A great circle can be drawn through any two points on a sphere.

19. If a spherical triangle is equilateral it is equiangular.

20. If the planes of two great circles of a sphere are perpendicular to each other, each circle passes through the poles of the other.

21. A spherical triangle can have only one right angle.

22. A zone on the earth's surface is bounded by two great circles.

23. The sides of a spherical triangle may be 40°, 20°, and 30°.

24. A circle midway between equator and the north pole is equal to half the equator.

25. If a plane is tangent to a sphere it contains all the lines tangent to the sphere at the point of tangency of the plane.

26. A sphere can be circumscribed about a right circular cylinder.

27. All points on a small circle of a sphere are equidistant from either of its poles.

28. The exterior angle of a spherical triangle is equal to the sum of the two non-adjacent interior angles.

29. The inscribed and circumscribed spheres of a tetrahedron are concentric.
EXERCISE IV  |  TEST-ANSWER

In each of the following exercises place a check mark before the expression that completes the sentence correctly.

1. A plane that is tangent to a sphere is

   (a) parallel to all radii
   (b) tangent to any radius
   (c) perpendicular to the radius of the sphere at the point of contact

2. Through four given points no three of which are in the same plane,

   (a) one and only one spherical surface can be passed
   (b) any number of spherical surfaces can be passed
   (c) no spherical surface can be passed

3. If a plane is passed through the center of a sphere the intersection of the plane and the spherical surface is

   (a) an oblong
   (b) an ellipse
   (c) a circle

4. The polar distance of a great circle of a sphere is an arc of

   (a) 60°
   (b) 90°
   (c) 180°

5. The sum of two sides of a spherical triangle is

   (a) less than the third side
   (b) equal to the third side
   (c) greater than the third side
6. The sum of the sides of a spherical polygon is
   (a) less than a great circle
   (b) equal to a great circle
   (c) $180^\circ$

7. The area of a spherical surface is equal to the area of
   (a) three great circles
   (b) one-third the area of four great circles
   (c) four great circles

8. The area of a zone is
   (a) $4\pi r^2$
   (b) $2\pi rh$
   (c) $4\pi rh$

9. The area of a spherical triangle expressed in spherical degrees is
   (a) equal to the spherical excess of the triangle
   (b) unequal to the spherical excess of the triangle
   (c) equal to one-half the spherical excess of the triangle

10. The volume of two spheres have the same ratio as
    (a) the cubes of their radii
    (b) the squares of their radii
    (c) their radii
EXERCISE V. POLAR TRIANGLES

1. In two polar triangles, each angle of the one is supplementary to the opposite side of the other.

Given two polar triangles ABC and XYZ.

To prove angle A + are YZ = 180°; angle B + are ZX = 180°; angle C + are YX = 180°.

Produce arcs AB and AC until they meet arc YZ in the points D and E.

Name the pole of arc AE.

What is the polar distance of a great circle?

Then arc YZ = 90°.

What is the pole of arc AD?

Will arc ZD = 90°?

Why?

Adding the two equations, we have YZ + EZ = 180°.

Why?

Dividing YZ into its component parts, we have YD + DE + EZ = 180°, or DE + YZ = 180°.

What is the spherical angle measured?

What is the measure of angle A?

If we substitute in the equation DE + YZ = 180°, we have angle A + are YZ = 180°.

What does angle X + are XG equal?

Can all other relations be proved in a similar way?

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Dr. David Eugene Smith, who wrote an introduction to her book, commented that this book revealed elementary mathematics "as a moving stream instead of a stagnant pool". Dr. Sanford uses the term elementary mathematics as closing with calculus.


Articles that are widely read and greatly appreciated by teachers of mathematics are included in this book. Many fine things in pedagogy that need to be available easily for mathematics teachers are found in this "source and guide book". The author translates the findings of educational research into classroom practice.


Several lectures were delivered by the author in 1913 before a club of graduate students of the University of Illinois on the subject of "The Philosophy of Mathematics". In 1913 this book was written containing these lectures.


A small but clearly stated book on the history of mathematics. It relates "our debt" to the Greeks and Romans for their development of mathematics.

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Volume I gives a general survey of the progress of elementary mathematics arranged by chronological periods with reference to racial and geographical conditions.

Volume II is for the purpose of supplying teachers and students with a usable textbook on the history of elementary mathematics through the first steps of calculus.
The Teaching of Elementary Mathematics, New York: Macmillan Co., 1916. Dr. Smith relates what the world has done in the way of making and of teaching mathematics, and what he feels is valuable literature of the subject. He clearly states some of the great questions of teaching.


ARSC, A. C., "Methods and Techniques of Teaching", California Journal of Secondary Education, XIV, No. 4, April 1939. The author discussed the methods and techniques used by teachers of the Los Angeles schools.

Beatty, Willard W., "Mathematics in the 20th Century", The Clearing House, XIII, April 1939. This article is an address given by Mr. Beatty at the National Council of Teachers of Mathematics on February 24 at Cleveland, Ohio. The author is a former president of the Progressive Education Association.

Blackhurst, J. H., "The Educational Value of Logical Geometry", The Mathematics Teacher, XXXII, April 1939. The title suggests the contribution this article makes. Five assumptions underlying the value and methods of teaching demonstrative geometry are listed by the author.

The author discusses how the central idea of re-vamping the curriculum to meet individual differences cannot be attained and then a solution is offered.

"Let's Face the Facts", The Mathematics Teacher, XXX, February 1937.

Data concerning enrollment in schools for various years are found in this article. Its purpose is to discuss the merits of any position which may be taken with respect to the place and content of mathematics in the high school curriculum.

Figure a. The Story of, Booklet, Detroit: The Burroughs Adding Machine Co., 1938.

The title suggests the contribution this booklet makes.


Supervisor Jarvis gives suggestions for improving the present condition concerning mathematics in the secondary schools.


A discussion by Mr. Lynch of Johann F. Herbert's philosophy of education as applied to mathematics.


This is a report of the sixteen lectures given at the National Council of Teachers of Mathematics held in Chicago.


A thesis revealing the history of mathematics to be used to enrich the junior high school curriculum.

The author discussed the problem of individual differences, the evils of mass education, the awful effect of emotional attitudes, the necessary qualifications of mathematics teachers, and the content of a general mathematics course.


The title suggests the contribution this article makes.


The title suggests the discussions made in this article.


About ten pages are given to various ways to vitalize plane geometry but many of her ideas apply to solid geometry.