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History of Mathematics as Enrichment of the Junior High School Mathematics Curriculum

Effie B. McDougall

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HISTORY OF MATHEMATICS AS ENRICHMENT OF THE JUNIOR HIGH SCHOOL MATHEMATICS CURRICULUM

This JUNIOR HIGH SCHOOL MATHEMATICS CURRICULUM growth of interest in attempts to make the mathematical studies more attractive to pupils who enter the junior high school grades. Curriculum revision has eliminated much of the old impractical arithmetic and substituted subject matter which can be linked with the problems of present day living. But there still remains the need to keep the present situations with past activity and demonstrate that unity of development which enlightens the awakening curiosity of the youthful mind. This need seems to be met by the stories of mankind's struggles to develop an adequate number system and the growth of mathematical processes to meet the requirements of expanding human interests. Why this form of enrichment has not been more extensively used in the greater share the psychological appeal.

A Dissertation Submitted in Partial Fulfillment of the Requirements for the Degree Master of Science in education of greater opportunity for the enthusiastic teacher who wishes to inspire young people with a real appreciation of their mathematical heritage from the past ages.

The author wishes to gratefully acknowledge the courtesy of Miss Grace Beedham, Principal of the Harrison School, Cleveland, Ohio, Mr. William Setz, Specialist in Mathematics by the Public Schools of Rochester, New York, Mr. Virgil Tewbough, Assistant Superintendent of the Indianapolis schools, and others who have so kindly answered letters of inquiry concerning questions relevant to this study. Also, helpful appreciation is accorded to Mrs. Georgia Lacey, Principal of Whittier School, Indianapolis, Ind., for her encouragement and helpful criticism during the course of its preparation.

COLLEGE OF EDUCATION
BUTLER UNIVERSITY
INDIANAPOLIS
1937
FOREWORD

This study has been the result of a gradual growth of interest in attempts to make the mathematical studies more attractive to pupils who enter the junior high school grades. Curriculum revision has eliminated much of the old impractical arithmetic and substituted subject matter which can be linked with the problems of present day living. But there still remains a need for something vital with human appeal to link the present situation with past activity and demonstrate that unity of development which delights the awakening curiosity of the youthful mind. This need seems to be met by the stories of mankind's struggles to develop an adequate number system and the growth of mathematical processes to meet the requirements of expanding human interests. Why this form of enrichment has not been more extensively used in the grades where its psychological appeal should be felt most keenly, is hard to say. The results of this study seem to indicate the beginning of such recognition and use of historical material by junior high school teachers and curriculum committees. May the future prove this trend to be in the direction of greater opportunity for the enthusiastic teacher who wishes to inspire young people with a real appreciation of their mathematical heritage from the past ages.

The author wishes to gratefully acknowledge the courtesy of Miss Grace Needham, Principal of the Harrison School, Lakewood, Ohio, Mr. William Betz, Specialist in mathematics for the Public Schools of Rochester, New York, Mr. Virgil Stinebaugh, Assistant Superintendent of the Indianapolis Schools, and others who have so kindly answered letters of inquiry concerning questions relevant to this study. Also, grateful appreciation is accorded to Mrs. Georgia Lacey, Principal of Whittier School, Indianapolis, Ind., for her encouragement and helpful criticism during the course of its preparation.

E. B. M.

Indianapolis, 1937
# TABLE OF CONTENTS

## PART TWO. THE COMPILOATION OF HISTORICAL MATERIAL WHICH CORRELATES WITH THE J.H.S. MATHEMATICS COURSE

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>I. INTRODUCTION</td>
<td>The need for curriculum enrichment in mathematics</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Purpose of the study</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>Plan of the study</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>The Junior High School defined</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>Psychological characteristics of the Junior high school pupil</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>Content of the J.H.S. mathematics curriculum</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>Meaning of enrichment of the curriculum</td>
<td>9</td>
</tr>
<tr>
<td></td>
<td>Sources of data</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>Summary</td>
<td>12</td>
</tr>
</tbody>
</table>

## PART ONE. THE PROBLEM

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>I. INTRODUCTION</td>
<td>The need for curriculum enrichment in mathematics</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Purpose of the study</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>Plan of the study</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>The Junior High School defined</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>Psychological characteristics of the Junior high school pupil</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>Content of the J.H.S. mathematics curriculum</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>Meaning of enrichment of the curriculum</td>
<td>9</td>
</tr>
<tr>
<td></td>
<td>Sources of data</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>Summary</td>
<td>12</td>
</tr>
</tbody>
</table>

## TABLE OF CONTENTS (CONTINUED)

<table>
<thead>
<tr>
<th>Item</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>FOREWORD</td>
<td>11</td>
</tr>
<tr>
<td>TABLE OF CONTENTS</td>
<td>11i</td>
</tr>
<tr>
<td>LIST OF TABLES</td>
<td>v1</td>
</tr>
</tbody>
</table>

## PART ONE. THE PROBLEM

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>THE USE OF HISTORICAL MATERIAL IN THE JUNIOR HIGH SCHOOL MATHEMATICS</td>
<td>13</td>
</tr>
<tr>
<td></td>
<td>Junior high school courses prior to 1930</td>
<td>13</td>
</tr>
<tr>
<td></td>
<td>State courses since 1930</td>
<td>14</td>
</tr>
<tr>
<td></td>
<td>City junior high school courses since 1930</td>
<td>15</td>
</tr>
<tr>
<td></td>
<td>Evaluation of outstanding courses for junior high school since 1930</td>
<td>20</td>
</tr>
<tr>
<td></td>
<td>Historical content of J. H. S. textbooks on mathematics since 1930</td>
<td>24</td>
</tr>
<tr>
<td></td>
<td>Evidences of the successful use of historical enrichment in J. H. S. mathematics</td>
<td>28</td>
</tr>
<tr>
<td></td>
<td>Summary</td>
<td>34</td>
</tr>
</tbody>
</table>

## PART ONE. THE PROBLEM

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>HISTORICAL MATERIAL WHICH CORRELATES WITH THE REGULAR J.H.S. MATHEMATICS</td>
<td>35</td>
</tr>
<tr>
<td></td>
<td>Such correlation exists and can be used effectively</td>
<td>35</td>
</tr>
<tr>
<td></td>
<td>Meager sources of reference material for pupil use</td>
<td>38</td>
</tr>
<tr>
<td></td>
<td>Basis for choice of correlating material which should be found in such a compilation</td>
<td>41</td>
</tr>
<tr>
<td></td>
<td>Summary</td>
<td>44</td>
</tr>
</tbody>
</table>
TABLE OF CONTENTS (CONTINUED)

PART TWO. THE COMPILATION OF HISTORICAL MATERIAL WHICH CORRELATES WITH THE J.H.S. MATHEMATICS COURSE

IV.--THE STORY OF NUMBER WORDS

<table>
<thead>
<tr>
<th>Topic</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Introduction</td>
<td>46</td>
</tr>
<tr>
<td>Counting among primitive tribes</td>
<td>48</td>
</tr>
<tr>
<td>Natural number words</td>
<td>50</td>
</tr>
<tr>
<td>Use of the fingers in counting</td>
<td>50</td>
</tr>
<tr>
<td>Compound number words among primitive tribes</td>
<td>51</td>
</tr>
<tr>
<td>Possibilities of our system of number words</td>
<td>52</td>
</tr>
</tbody>
</table>

V.--EARLY NUMBER SYMBOLS AND SYSTEMS OF NOTATION

<table>
<thead>
<tr>
<th>Topic</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Picture numbers</td>
<td>54</td>
</tr>
<tr>
<td>Natural numbers</td>
<td>55</td>
</tr>
<tr>
<td>Early number systems</td>
<td>56</td>
</tr>
<tr>
<td>The Hindu-Arabic numerals with zero</td>
<td>69</td>
</tr>
</tbody>
</table>

VI.--NUMBER THEORIES AND OLD METHODS OF CALCULATION

<table>
<thead>
<tr>
<th>Topic</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Meaning of arithmetic in the past</td>
<td>72</td>
</tr>
<tr>
<td>Number superstitions</td>
<td>73</td>
</tr>
<tr>
<td>Kinds of numbers studied by the Greeks</td>
<td>75</td>
</tr>
<tr>
<td>The magic square</td>
<td>76</td>
</tr>
<tr>
<td>Finger reckoning</td>
<td>78</td>
</tr>
<tr>
<td>The abacus and counting board</td>
<td>79</td>
</tr>
</tbody>
</table>

VII.--TYPES OF NUMBERS AND THEIR USE IN CALCULATION

<table>
<thead>
<tr>
<th>Topic</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Negative numbers</td>
<td>84</td>
</tr>
<tr>
<td>Irrational numbers</td>
<td>84</td>
</tr>
<tr>
<td>Unit fractions</td>
<td>86</td>
</tr>
<tr>
<td>Sexagesimal fractions</td>
<td>87</td>
</tr>
<tr>
<td>Decimal fractions</td>
<td>89</td>
</tr>
<tr>
<td>Per cent fractions</td>
<td>90</td>
</tr>
<tr>
<td>Algorithm and the evolution of the four processes</td>
<td>93</td>
</tr>
</tbody>
</table>

VIII.--PRACTICAL APPLICATIONS OF MATHEMATICAL CALCULATIONS

<table>
<thead>
<tr>
<th>Topic</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Astronomy fostered by religion</td>
<td>100</td>
</tr>
<tr>
<td>Commercial practices in ancient Babylonia</td>
<td>102</td>
</tr>
<tr>
<td>Greek and Phoenician commercial practices</td>
<td>103</td>
</tr>
<tr>
<td>Practical mathematics in Egypt</td>
<td>103</td>
</tr>
<tr>
<td>Civil engineering among the Romans</td>
<td>105</td>
</tr>
<tr>
<td>Demoralized commerce of the Dark Ages</td>
<td>105</td>
</tr>
<tr>
<td>Interest and banking practices</td>
<td>107</td>
</tr>
<tr>
<td>Partnerships for investment purposes</td>
<td>111</td>
</tr>
<tr>
<td>Taxation</td>
<td>111</td>
</tr>
</tbody>
</table>
# TABLE OF CONTENTS (CONTINUED)

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>I.</td>
<td>Insurance</td>
<td>112</td>
</tr>
<tr>
<td></td>
<td>Forms of money</td>
<td>114</td>
</tr>
<tr>
<td></td>
<td>Measurement of time and distance</td>
<td>115</td>
</tr>
<tr>
<td>II.</td>
<td>IX.--THE EVOLUTION OF ALGEBRA</td>
<td>121</td>
</tr>
<tr>
<td></td>
<td>The oldest record of algebra</td>
<td>121</td>
</tr>
<tr>
<td></td>
<td>Old problems</td>
<td>122</td>
</tr>
<tr>
<td></td>
<td>Diophantus, the father of algebra</td>
<td>126</td>
</tr>
<tr>
<td></td>
<td>Origin of the word, algebra</td>
<td>127</td>
</tr>
<tr>
<td></td>
<td>Contribution of Fibonacci</td>
<td>129</td>
</tr>
<tr>
<td></td>
<td>Symbolism developed in the seventeenth century</td>
<td>129</td>
</tr>
<tr>
<td></td>
<td>Use of algebra in the future</td>
<td>131</td>
</tr>
<tr>
<td>III.</td>
<td>PART THREE. THE CONCLUSION</td>
<td>132</td>
</tr>
<tr>
<td></td>
<td>X.--INTUITIVE GEOMETRY AND INDIRECT MEASUREMENT</td>
<td>132</td>
</tr>
<tr>
<td></td>
<td>Intuitive geometry learned from natural world</td>
<td>132</td>
</tr>
<tr>
<td></td>
<td>Use of geometry in ornamentation</td>
<td>132</td>
</tr>
<tr>
<td></td>
<td>Its use in the form of primitive shelters</td>
<td>134</td>
</tr>
<tr>
<td></td>
<td>First geometric measurements</td>
<td>135</td>
</tr>
<tr>
<td></td>
<td>The rope stretchers of Egypt</td>
<td>135</td>
</tr>
<tr>
<td></td>
<td>Thales of Miletus</td>
<td>136</td>
</tr>
<tr>
<td></td>
<td>The brotherhood of Pythagoras</td>
<td>137</td>
</tr>
<tr>
<td></td>
<td>Alexandrian scholars</td>
<td>139</td>
</tr>
<tr>
<td></td>
<td>Beginnings of trigonometry</td>
<td>142</td>
</tr>
<tr>
<td></td>
<td>Conclusion</td>
<td>143</td>
</tr>
<tr>
<td>IV.</td>
<td>XI.--CONCLUSIONS AND RECOMMENDATIONS</td>
<td>144</td>
</tr>
<tr>
<td></td>
<td>Conclusions</td>
<td>145</td>
</tr>
<tr>
<td></td>
<td>Recommendations</td>
<td>147</td>
</tr>
<tr>
<td>V.</td>
<td>BIBLIOGRAPHY</td>
<td>149</td>
</tr>
</tbody>
</table>
### Table of Contents

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>I.</td>
<td>General and Historical Content of Fourteen J.H.S. Courses of Study in Mathematics Issued Since 1930</td>
<td>16</td>
</tr>
<tr>
<td>II.</td>
<td>Summary of Historical Content of Fourteen J.H.S. Mathematics Courses Issued Since 1930</td>
<td>20</td>
</tr>
<tr>
<td>III.</td>
<td>Historical Content of Seventeen J.H.S. Textbook Series on Mathematics Published Since 1930</td>
<td>25</td>
</tr>
</tbody>
</table>

**Introduction**

The trend of educational methods at the present time is away from the old idea of forcible feeding and toward the creation of a wholesome hunger for knowledge. In the fields of literature, social studies, and science this trend has been particularly marked. Evidence of it may be found in the enriched curricula for elementary and junior high school grades. Many suggested types of activity offer an opportunity for individual pupil research. Primitive life with its many hardships is contrasted with modern modes of living by means of building activities, dramatization and stories. Teachers and pupils dramatize the daily life in other lands, reproduce its setting in colorful scenes, build ancient castles where the imagination may picture the knights and ladies of old, paint again the pictures of primitive peoples,
HISTORY OF MATHEMATICS AS ENRICHMENT OF THE 
JUNIOR HIGH SCHOOL MATHEMATICS CURRICULUM 

PART ONE. THE PROBLEM

CHAPTER I

INTRODUCTION

The trend of educational methods at the present time is away from the old idea of forcible feeding and toward the creation of a wholesome hunger for knowledge. In the fields of literature, social studies, and science this trend has been particularly marked. Evidence of it may be found in the enriched curricula for elementary and junior high school grades. Many suggested types of activity offer an opportunity for individual pupil research. Primitive life with its many hardships is contrasted with modern modes of living by means of building activities, dramatization and stories. Teachers and pupils dramatize the daily life in other lands, reproduce its setting in colorful scenes, build ancient castles where the imagination may picture the knights and ladies of old, paint again the pictures of primitive people,
and interpret their myths and folk songs. The exciting lore of communication and transportation development is interpreted to the youth of today through varied and vitalizing activities. The history of civilization is emphasized through the many human interest stories which surround the great inventions and discoveries of science.

But these stimulating activities have not found their way into many mathematics classrooms. Mathematics courses have remained comparatively barren of such enrichment as might vitally link the subject with the drama of human endeavor. Mathematics has been termed a skill subject, worth while only from the standpoint of utility. Romance has had no place in its presentation. Yet the efforts of the human race to express the number concept have kept pace with the growth of written language and, inseparably, they measure the strides of civilization.

History of mathematics is the history of number expression among primitive peoples and its gradual development from the simplest forms into the thoroughly efficient processes and symbolism of modern mechanics, science, and commerce. It is the history of form and measurement, with the relationship thereof; the story of men with intellectual curiosity to find and explore the possibilities of such relationships. It is also the recognition of the individuals and races which have contributed greatly to the sum total of mathematical knowledge.
and its practical applications. Therefore, this study entitled, "History of Mathematics as Enrichment of the Junior High School Curriculum" seems to be justified by the very nature of the content just mentioned. Why should not the awakening curiosity of the pre-adolescent boy or girl find in the history of mathematics, a new source of fascinating pioneer stories, if such information were available in an attractive form and diction suited to his limited background of experience?

The Purpose of the Study

The purpose of this study grew out of the serious consideration of the question just stated, together with the actual experience of attempting such enrichment of mathematics in the junior high school classroom. The motivating question might be stated thus: Have we neglected historical material correlating with the junior high school course of study in mathematics which would enrich and vitalize the general content? The premise upon which the study is based might be summed up in these two sentences: Enrichment of the mathematics course of study for the junior high school is desirable and new sources should be sought and utilized; history of mathematics has ample material which might lend enrichment through the human interest appeal in man's struggle to learn the laws of the universe. The study will endeavor to prove the existence of historical material which
can be correlated with the junior high school course of study; to ascertain the extent of its present use in junior high school curricula and textbooks; to evaluate the experience of those who have used such historical material; and to summarize conclusions and recommendations concerning its possible use in the future.

The Plan of the Study

The first step in seeking an answer to the motivating question just stated was a survey of junior high school courses of study, textbooks, and current magazine articles concerning the mathematics curriculum to determine what historical material has been used or is being used at the present time. The results of this survey are given in detail in the following chapter. The second step was a search for material now available on the history of mathematics which could be used by the junior high school pupil himself. Since this effort revealed a dearth of such material, the final step was a study of mathematical history to prove that sufficient historical material can be made available for such enrichment without introducing that which is irrelevant to the background of mathematics usually taught in the elementary and junior high school courses. The result of this phase of the study is the compilation which is set forth in the various chapters of Part Two. The conclusions and recommendations resulting from the study will be found in the concluding chapter.
The Junior High School Defined

For many years the seventh and eighth grades were included in the elementary school, and the secondary or high school began with the ninth grade. During the last quarter of a century, this division has been rapidly shifting to a three unit organization, consisting of the elementary school, the junior high school, and the senior high school. The elementary school includes the first six grades under the new division, while the seventh, eighth and ninth grades comprise the junior high school. Generally speaking, the junior high school personnel is the pre-adolescent age group of children from eleven to fourteen inclusive. The junior high school pupil, when referred to in this study, is considered as one of this age group with its usual psychological characteristics. Since these psychological urges of the junior high school pupil necessarily concern the study of interest provoking curriculum enrichment, we need to note a few of the more outstanding.

Psychological Characteristics of the Junior High School Pupil

A child of junior high school age is first of all overwhelmed with intellectual curiosity concerning the origin of everything with which he comes into contact. Tracy says, "There is a maximum of enthusiastic interest in things." ¹

did these ideas originate? Who first made such things as are now common? What did people use before they had modern inventions? Many similar questions occur to the alert and vigorous mind of the pre-adolescent child. Another normal urge of such a child is the desire to read and investigate a subject on his own responsibility. "Children now become more disposed to undertake things for themselves and without assistance from others." The child of this age also becomes intensely interested in people and in the things people have done. Bits of biographical sketches which tell of unusual ability and courage in the face of discouragement, never fail to interest him. Again quoting from Tracy,

"In youth this native instinct [curiosity] becomes developed into a keen and active interest in all manner of things, including . . . the products of the inventive genius of man."

These recognized urges of pre-adolescent youth certainly indicate that in the realm of education, the junior high school child should be encouraged to investigate, criticize and explore all the ways by which mankind has emerged from the primitive state. The development of mathematics and the lives of great men who have been responsible for it, should certainly receive a reasonable share of attention in such a program of investigation and activity.

2 Ibid., Ch. II, p. 13.
3 Ibid., Ch. V, p. 67.
The story of the abacus along with the actual use of a Chinese abacus or suanpan in the classroom has been found more effective in fixing the idea of place value of our numerals than any other method of presentation. The story of the rope stretchers of Egypt and their desire to orient their temples, effectively motivated by actual experience in knotting such a rope, proved the 3:4:5 ratio of the right triangle in a way that no amount of mere factual teaching could have done. The stories of men whose lives were devoted to the painstaking work of writing mathematical books in manuscript, that the knowledge already accumulated might not be lost to the world, have been found to inspire the interest of pupils in the beginnings of algebra and geometry. Puzzle problems and number superstitions have enlivened many class periods and afforded the opportunity for calling attention to the more desirable and practical applications of mathematics to be found in modern textbooks and courses of study.

The Content of Junior High School Mathematics Curricula

Under the old two unit or eight-four system of organization of the public schools, there was a decided difference between the eighth and ninth grades, not only in classroom organization and management but also in curriculum content. Departmentalization of the seventh and eighth grades was the
first corrective step in the attempt to overcome this weakness. It helped the pupil to adapt himself more readily to high school classroom organization but did very little to bridge the wide differences between the elementary and high school curricula. This change from one to the other quite often proved to be a stumbling block for the bewildered high school freshman. The organization of the junior high school has changed this situation by introducing traditional high school subjects through short exploratory units much earlier in the general course and by developing self-directed study through multiple choice activities and individual assignments. Of the three junior high school grades, the ninth grade has experienced the least change in curriculum content, but the curriculum of the seventh and eighth grades has undergone a radical metamorphosis under the new plan. It has been revised to include much subject matter which has hitherto been sacred to the senior high school. In the mathematics course, this new material includes intuitive geometry, an inkling of trigonometry through indirect measurement and the tangent ratio, elementary algebra and forms of graphing. Along with this new material, the more practical phases of business and social arithmetic have been retained while those, now obsolete, have been dropped. Frequent review of the fundamental processes with practice in the reading and writing of numbers is given through meaningful activities and forms of self
improvement competition. This general summary of content was obtained by the examination of the various junior high school courses of study issued since 1930. A more detailed account of individual units found in these courses will be given in the following chapter.

Meaning of Enrichment of the Curriculum

Enrichment of the curriculum as used in this study should be considered as the inclusion of anything pertaining to a subject, which by its introduction, will tend to make more attractive, the process of acquiring the fundamental skills and knowledge necessary to a liberal education. Enrichment may take the form of additional information obtained by individual research and presented in an oral or written report or it may be found in constructive activities, dramatization, graphic expression, or story telling. It must be in any case something which can increase pupil interest and broaden his appreciation and understanding of the subject to be enriched. Enrichment must vitalize and not deaden the effects of former teaching. Such results can only be determined by the experience of those who have tried and tested forms of potential enrichment in the classroom.

History of mathematics as enrichment material in the junior high school classroom has not been the subject of many recorded experiments, as further developments in this study will endeavor to show, but such testimonials as we have, the
indicate favorable results. Until very recently history of mathematics as a subject for study was considered suitable only for the university student. While we grant this opinion to be true insofar as the entire scope of the subject is concerned, we feel reasonably sure from even the meager use it has had up to the present time, that certain phases of historical information have fitted into or may fit into the pattern now being followed in the junior high school mathematics courses.

Sources of Data

The sources of historical material for the compilation to be given in Part Two of this study were the many excellent mathematical histories in our libraries, a complete list of which is found in the bibliography. These histories with few exceptions are not within the reading ability of the average junior high school pupil. Even if he possessed the vocabulary necessary to read the words, he would soon be lost in the maze of content for which he has no background or preparation. The chapters of the compilation have not been a work of research for new material but rather a search in the ocean of existing material for the few pearls which might prove to be of great price to the young student of a subject traditionally considered as dull and devoid of romantic interest.

The sources of data concerning curriculum content were the courses of study issued by city school systems which have recognized the junior high school as a distinct unit in the
public school organization. Earlier courses have been examined, but only those issued since 1930 have been included in the tabulation of content found in Chapter II. The United States Circular No. 139 issued by the Office of Education, United States Department of the Interior, was the source from which the list of courses for junior high schools issued since 1930 was obtained. Courses which have been described as outstanding among the entire list were thus evaluated either by the Research Division of the National Educational Association or by the Curriculum Bureau of Teachers College, Columbia University. These evaluations were made upon request and the information given has been quoted from personal letters. Information concerning the actual use of historical material in junior high school teaching is quoted from personal letters by permission of the authors.

The textbooks tabulated are those which have been definitely indicated as junior high school books or those prepared for one or more of the grades included in the junior high school. They, also are limited to those which have been published during the past six years. Such books, not in the Teachers Special Library of Indianapolis, have been obtained from Chicago libraries in connection with the University of Chicago. These

12.

Libraries were consulted in the search for available material on the history of mathematics suitable for supplementary reading by the junior high school pupil and inquiry for such material was also made of the Research Division of the N.E.A. The results of this survey will be given in a later chapter. Books and periodicals which have been consulted in any part of this study have been listed in the bibliography and those giving specific help will be referred to from time to time in the footnotes.

**Summary**

Thus far we have endeavored to set forth the present trends of educational thought as expressed in the junior high school curricula with particular emphasis on the content of the courses in mathematics. Materials suitable for enrichment in this particular field from the history of mathematics have been indicated as the subject of this study with the question to be considered as follows: Have we neglected historical material correlating with the junior high school course of study in mathematics which would enrich and vitalize the general content? The meaning of the term junior high school has been defined as to grade level and the changes of curricula in these grades have been noted. In order to more fully set forth the purpose of the study, the terms enrichment and history of mathematics have been defined. The sources of data have been briefly indicated and the general plan of the study discussed. We can now proceed to a more detailed account of the survey and its results.
CHAPTER II

THE USE OF HISTORICAL MATERIAL IN JUNIOR HIGH SCHOOL MATHEMATICS

Use of Historical Material in Junior High School Courses before 1930

So far as it was possible to ascertain from the files, the use of historical material in the junior high school curriculum was not mentioned definitely in magazine articles prior to 1930, nor was it definitely included in any course of study before the present decade. Before 1930, separate and distinct courses of study for the junior high school had been issued by only a few of the larger city school systems. Several of these older courses were located and examined although no attempt at an exhaustive survey was made. Four representative cities, namely, Detroit, Chicago, Kansas City

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2 Chicago, Ill. Course of Study in Mathematics for Junior High Schools, 1929.
3 Kansas City, Mo. Public Schools Course of Study in Mathematics for Junior High School Grades, 1929.
grades in any way segregated; and only two\(^7\) of these actually named the junior high school as a distinct unit. In Indiana it was included with the elementary grades as Grades 1 to 9, and in Iowa it was included in the high school as Grades 7 to 12. The general content of these state courses varied from city courses only in minor details but no mention of history of mathematics as enrichment was found in them. Thus, the state courses are practically eliminated from a study of junior high school curricula, and entirely eliminated as a source of information concerning the use of historical material as enrichment.

The Use of History of Mathematics in City Courses for the Junior High School since 1930

For the year 1930, Circular No. 139 lists seven city courses which had been outlined for one or more of the junior high school grades. Not one of the seven, however, are listed as distinctly junior high school courses. The content of these courses includes some of the modern subject matter such as elementary topics of algebra, intuitive geometry and arithmetic of the home and community. Some are merely formal outlines, others more fully developed in typical units, but none make any use of historical material in connection with the units.

Since 1930, curriculum revision has progressed rapidly. Fourteen outlined courses in mathematics for junior high school or junior high school grades have appeared. This list includes

\(^7\)Minnesota and Idaho.
only city school systems that have recognized the distinctive grouping of the seventh, eighth and ninth grades as a secondary unit. In some courses only seventh and eighth are included, with a separate course for ninth grade. In that type of outline the seventh and eighth grade course has been indicated in the survey as being more representative of the new curricula. The purpose of this survey of junior high school courses is twofold. It is, first, to indicate the use made of historical background for the various subjects of the mathematics curriculum, and, second, to show the general content of these courses with which history of mathematics might be correlated. The tabulation form, being the most concise, has been used to indicate the results of this survey.

**TABLE I. GENERAL AND HISTORICAL CONTENT OF FOURTEEN J.H.S. COURSES OF STUDY IN MATHEMATICS ISSUED SINCE 1930.**

<table>
<thead>
<tr>
<th>City</th>
<th>Year</th>
<th>Grades</th>
<th>Pp.</th>
<th>General Content</th>
<th>Amount and Types of Historical Material Suggested</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chicago</td>
<td>1933</td>
<td>7-9</td>
<td>134</td>
<td>Arithmetic, algebra, intuitive geometry</td>
<td>No historical material mentioned</td>
</tr>
<tr>
<td></td>
<td>Revised from 1929</td>
<td></td>
<td></td>
<td></td>
<td>To suggestions of historical material. The short reference to origin of angle.</td>
</tr>
<tr>
<td>Denver</td>
<td>1931</td>
<td>7-9-8</td>
<td></td>
<td>Work book type by grades, business and home arithmetic, algebra, mensuration</td>
<td>No historical material indicated</td>
</tr>
</tbody>
</table>
TABLE I. GENERAL AND HISTORICAL CONTENT OF FOURTEEN J.H.S. COURSES OF STUDY IN MATHEMATICS ISSUED SINCE 1930. 17.

<table>
<thead>
<tr>
<th>City</th>
<th>Year Revised</th>
<th>Grades</th>
<th>Fp.</th>
<th>General Content</th>
<th>Amount and Types of Historical Material Suggested</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fort Worth, Texas</td>
<td>1936</td>
<td>7-9</td>
<td>198</td>
<td>Development of quantitative thinking through arithmetic of everyday life.</td>
<td>Definite activities suggested in history of mathematics on twenty-three different pages. Correlated with thirteen of the twenty-one units outlined.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Algebra I and II. Very well organized.</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Carefully outlined.</td>
<td></td>
</tr>
<tr>
<td>Indianapolis, Ind.</td>
<td>1933</td>
<td>7-9</td>
<td>162</td>
<td>Business and home practices in arithmetic, intuitive geometry, algebra.</td>
<td>Unit on history of mathematics suggested but not outlined in 1933. Tentative course. Not given in 1934 revision.</td>
</tr>
<tr>
<td></td>
<td>1934</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lake-wood, Ohio</td>
<td>1935</td>
<td>7-8</td>
<td>117</td>
<td>Business and home arithmetic, algebra, intuitive geometry, graphing.</td>
<td>Six pages devoted to a definitely outlined unit on history of mathematics. Long list of suggested activities.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Los Angeles, Calif.</td>
<td>1933</td>
<td>7-9</td>
<td>95</td>
<td>Arithmetic, Household problems, Brief introduction to algebra and geometry.</td>
<td>Two pages devoted to suggestions for use of history as introduction to review of fundamentals. Fp. 11 and 12.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Louisville, Ky.</td>
<td>1933</td>
<td>7-8</td>
<td>-</td>
<td>Arithmetic, Measurement Formulas.</td>
<td>No suggested use of historical material. One short reference to origin of angle measurement, p. 15.</td>
</tr>
</tbody>
</table>
TABLE I. GENERAL AND HISTORICAL CONTENT OF FOURTEEN J.H.S. COURSES OF STUDY IN MATHEMATICS ISSUED SINCE 1930.

<table>
<thead>
<tr>
<th>City</th>
<th>Year</th>
<th>Grade</th>
<th>Pp.</th>
<th>General Content</th>
<th>Amount and Types of Historical Material Suggested</th>
</tr>
</thead>
<tbody>
<tr>
<td>Muncie Ind.</td>
<td>1932</td>
<td>7-9</td>
<td></td>
<td>Arithmetic of the community, Practical applications of algebra, geometry and trigonometry</td>
<td>Suggested use of historical material through biographical study of ancient and modern mathematicians. Pp. 13, 35 and 45</td>
</tr>
<tr>
<td>Newark, N.J.</td>
<td>1932</td>
<td>7-8</td>
<td></td>
<td>Arithmetic, Intuitive geometry, Formulas</td>
<td>No mention of historical material</td>
</tr>
<tr>
<td>New Brunswick, N.J.</td>
<td>1933</td>
<td>7-12 Jr. and Sr. H.S.</td>
<td>120</td>
<td>Arithmetic, algebra, geometry, introductory trigonometry</td>
<td>No mention of historical material in actual course. One of aims given in foreword suggests its possible use in giving cultural background</td>
</tr>
<tr>
<td>Rochester, N.Y.</td>
<td>1932</td>
<td>7-9</td>
<td>149</td>
<td>Applied arithmetic, algebra, intuitive geometry, graphs</td>
<td>No mention of historical material in the course of study but the Wm. Betz textbooks which are followed in Rochester use it extensively.</td>
</tr>
</tbody>
</table>
(Continued)

### TABLE I. GENERAL AND HISTORICAL CONTENT OF FOURTEEN J.H.S. COURSES OF STUDY IN MATHEMATICS ISSUED SINCE 1930.

<table>
<thead>
<tr>
<th>City</th>
<th>Year</th>
<th>Grades</th>
<th>Pp.</th>
<th>General Content</th>
<th>Amount and Types of Historical Material Suggested</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sacramento, Calif.</td>
<td>1932</td>
<td>7-9</td>
<td>72</td>
<td>Arithmetic, algebra, intuitive geometry</td>
<td>Use of historical material mentioned in aims, p.8 and general methods, p. 13. Urges encouragement of pupils to look up interesting historical topics. Not mentioned later in body of the course.</td>
</tr>
<tr>
<td>Washington, D.C.</td>
<td>1933</td>
<td>7-9</td>
<td>6</td>
<td>Mere topical outline with time allotments. Topics in arithmetic, algebra and measurements</td>
<td>No mention of historical material</td>
</tr>
</tbody>
</table>

Of the fourteen courses outlined in the above table, seven contain no mention of historical material, three mention it in the foreword or list of aims, three indicate activities involving its use with one or more units, and one has an outlined unit on historical material. These facts are summarized in the following tables.
TABLE II. SUMMARY OF HISTORICAL CONTENT OF FOURTEEN J.H.S. MATHEMATICS COURSES ISSUED SINCE 1930

<table>
<thead>
<tr>
<th>Group</th>
<th>Use of Historical Material</th>
<th>No.</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>Outlined unit</td>
<td>1</td>
</tr>
<tr>
<td>II</td>
<td>Suggested in activities</td>
<td>3</td>
</tr>
<tr>
<td>III</td>
<td>Suggested in foreword</td>
<td>3</td>
</tr>
<tr>
<td>IV</td>
<td>Not mentioned</td>
<td>7</td>
</tr>
</tbody>
</table>

Evaluation of Outstanding Courses in the Tabulated List found in Table I

In order to interpret the significance of this survey more fully, well known authorities in the curriculum field were asked to name the courses which were considered the most outstanding of those issued since 1930. The Curriculum Bureau of Teachers College Columbia University answered this request as follows:

Among the courses of study in junior high school mathematics published since 1930 the following have been considered to be worthy pieces of work by the Curriculum Bureau:

Denver, Col.--Mathematics--Junior High School--1931.
Fort Worth, Texas--Mathematics--A tentative course for junior high schools. Grades 7-9, 1936.
Lakewood, Ohio--Mathematics--A tentative Course of Study for Junior High Schools, 1935.
Muncie, Indiana--Tentative Course of Study in Junior High School Mathematics, 1932.
In Rochester, New York--Tentative Course of Study in Junior High School Mathematics, 1932.


The following reply to a similar request sent to the Research Bureau of the National Educational Association was received:

1/ Outstanding Courses of Study in Mathematics

<table>
<thead>
<tr>
<th>City and State</th>
<th>Title</th>
<th>Date</th>
</tr>
</thead>
<tbody>
<tr>
<td>Indianapolis, Ind.</td>
<td>Course of Study in Mathematics Grades 7, 8 and 9</td>
<td>1934</td>
</tr>
<tr>
<td>Lakewood, Ohio</td>
<td>Mathematics--A Tentative Course for Junior High Schools</td>
<td>1935</td>
</tr>
<tr>
<td>Muncie, Ind.</td>
<td>Tentative Course of Study in J.H.S. Mathematics Grades 7-9</td>
<td>1932</td>
</tr>
<tr>
<td>New Brunswick, N.J.</td>
<td>Mathematics Course of Study for Junior and Senior High Schools</td>
<td>1933</td>
</tr>
<tr>
<td>Passaic, N. J.</td>
<td>Course of Study in Mathematics Grades 7-8</td>
<td>1935</td>
</tr>
</tbody>
</table>

Source: Taken from reports of the Society for Curriculum Study of which Henry Harap, Western Reserve University, Cleveland, Ohio, is chairman.

8 Quoted from list sent from Curriculum Bureau, Teachers College Columbia University, Dec. 1, 1936, by C. B. Upton, Professor of Mathematics.

9 Quoted from list sent from Research Division of the National Educational Association November 25, 1936, by Associate Director, Frank W. Hubbard.
In the foregoing evaluations of junior high school courses only two courses were cited by each of the authors. These were Lakewood, Ohio, and Muncie, Ind. It is extremely significant to this study that Lakewood is the one course out of fourteen that shows a completely outlined unit on the history of mathematics, and Muncie is one of the group indicating its extensive use through suggested enrichment activities. The Fort Worth course is not mentioned by the N.E.A. Research Bureau because it is a 1936 edition and their list includes only 1930-1935 courses. It is a very well organized course and doubtless will be included as outstanding in later valuations.

Of the nine different courses mentioned in the two lists, only two are found in Group IV of Table II on page 20. These are Indianapolis, Indiana, and Rochester, New York.

Seeking a reason for this, communications were sent to the Curriculum Administration Department of each city asking why historical content had been omitted. Indianapolis was found to have included a suggestion for such a unit in the first tentative course of 1933 but the unit was not definitely organized. In the 1934 revision, it was omitted entirely. Mr. Virgil Stinebaugh, assistant superintendent of the Indianapolis Schools in charge of Junior High School and Curriculum Studies indicated the attitude of the committee as follows:
Our committee thought that it was not the type of subject which should be developed as a separate unit. It was our feeling that the history of mathematics should be brought in in relation to each of the units.¹⁰

He expressed his personal attitude toward such enrichment in these words: "I am very much in favor of this subject being given considerable emphasis."¹¹ A special bulletin containing suggestive material on the historical phases of mathematics was sent to junior high school teachers of Indianapolis by Mr. Stinebaugh's office last year. The attitude of the Indianapolis school system is thus indicated as favorable toward this form of enrichment but undecided as to its method of introduction.

Mr. William Betz, head of the Mathematics Department in the Rochester, New York, schools, is, himself, the author of a very modern series of textbooks for both junior and senior high schools. In reply to the letter of inquiry sent to Rochester, Mr. Betz said that his books were used as textbooks throughout their system and since they included abundant historical material, it was not considered necessary to outline it as subject matter in the course of study. He is decidedly in favor of its use as enrichment but believes it

¹⁰Quoted by permission of the author, Mr. Virgil Stinebaugh, from communication of March 8, 1937.

¹¹Ibid.

¹²Quoted by permission of the author, Mr. William Betz, communication of March 10, 1937.
should be taught as introductory approach to the various phases of arithmetic and higher mathematics. His personal attitude he expressed as follows:

In all my own teaching practically from the beginning of my career, I have invariably linked my work to the historical soil from which mathematics sprung. I think that I am also correct in saying that the majority of our Rochester teachers of mathematics have followed my example to a very large extent.\textsuperscript{12}

Thus Rochester is seen to be very much in favor of historical enrichment and actually making extensive use of it despite the fact that the course of study does not mention it. Historical Content of J. H. S. Textbooks on Mathematics Published since 1930.

A survey of recent junior high school textbooks indicates that Mr. William Betz of Rochester has led the way to the use of historical material in the junior high school textbook. His Junior Mathematics for Today, Books I and II contain twenty-four pages or partial pages devoted to historical discussion and twenty-six pictures relating to it. It is used as a natural part of the approach to a lesson, with story and picture adding the interest incentive to its study. Other textbooks of very recent date show a significant trend toward the inclusion of historical information but as yet it seems to be largely in an experimental stage. While no claim to

\textsuperscript{12}Quoted by permission of the author, Mr. William Betz, from communication of March 12, 1937.
an exhaustive survey is made, the field has been fairly well covered in the list of textbooks examined. These books are listed in the following table, with the extent of historical content indicated by the number of pages devoted to it.

**TABLE III. HISTORICAL CONTENT OF SEVENTEEN J. H. S. TEXTBOOK SERIES ON MATHEMATICS PUBLISHED SINCE 1930.**

<table>
<thead>
<tr>
<th>Name of Textbook</th>
<th>Authors</th>
<th>Publisher and Date of Publication</th>
<th>Historical Content by Pages</th>
</tr>
</thead>
<tbody>
<tr>
<td>The Problem and Practice Arithmetic Third Book</td>
<td>Smith-Luse-Morse</td>
<td>Ginn and Co., Boston 1930</td>
<td>None</td>
</tr>
<tr>
<td>Unit Mastery Arithmetic Advanced Book</td>
<td>Stone-Mills</td>
<td>Benj. H. Sanford Co., Chicago 1930</td>
<td>None</td>
</tr>
<tr>
<td>Arithmetic for Today Book III</td>
<td>Anderson-Gade</td>
<td>Silver-Burdeet Co., N. Y. 1931</td>
<td>None</td>
</tr>
<tr>
<td>Mathematics for Junior High Schools, Triangle Series Books I and II</td>
<td>Brueckner, Anderson and Banting</td>
<td>John C. Winston Co., Philadelphia 1931</td>
<td>None</td>
</tr>
<tr>
<td>Iroquois Arithmetics</td>
<td>DeGreat, Firman, Smith</td>
<td>Iroquois Pub., Co., Syracuse, N.Y. 1932</td>
<td>None</td>
</tr>
<tr>
<td>The New Day Arithmetic Books I and II</td>
<td>Durell, Foberg, Newcomb</td>
<td>Chas.E.Merrill, Chicago 1932</td>
<td>None</td>
</tr>
<tr>
<td>Name of Textbook</td>
<td>Authors</td>
<td>Publisher and Date of Publication</td>
<td>Historical Content by Pages</td>
</tr>
<tr>
<td>------------------</td>
<td>--------</td>
<td>---------------------------------</td>
<td>----------------------------</td>
</tr>
<tr>
<td>The New Day Arithmetic Book III</td>
<td>Vevia Blair</td>
<td>Chas. E. Merrill, Chicago 1933</td>
<td>Six notes and four pictures</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Book II, Eleven pages and nine pictures on historical background</td>
</tr>
<tr>
<td>A First Course in the New Mathematics</td>
<td>Edgerton and Carpenter</td>
<td>Allyn Bacon Boston 1934</td>
<td>Introductory article on &quot;Beginnings of Mathematics, four pages</td>
</tr>
<tr>
<td>A Second Course in the New Mathematics</td>
<td>Edgerton and Carpenter</td>
<td>Allyn Bacon Boston 1934</td>
<td>One article concerning Pythagoras</td>
</tr>
<tr>
<td>A Third Course in the New Mathematics</td>
<td>Edgerton and Carpenter</td>
<td>Allyn Bacon Boston 1934</td>
<td>One article on p. 324</td>
</tr>
<tr>
<td>Mastery Arithmetic Book II</td>
<td>Bodley, Gibson, Hayes and Watson</td>
<td>D. C. Heath and Co. N. Y. 1934</td>
<td>One picture</td>
</tr>
<tr>
<td>Modern School Mathematics Book I and II</td>
<td>Schorling Clark and Smith</td>
<td>World Book Co. Yonkers on Hudson 1935</td>
<td>Book I, None</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Book II, Two page article on history of numerals and abacus with several pictures</td>
</tr>
</tbody>
</table>
### TABLE III. HISTORICAL CONTENT OF SEVENTEEN J.H.S. TEXT-BOOK SERIES ON MATHEMATICS PUBLISHED SINCE 1930.

<table>
<thead>
<tr>
<th>Name of Textbook</th>
<th>Authors</th>
<th>Publisher and Date of Publication</th>
<th>Historical Content by Pages</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mathematics at Work</td>
<td>George H. Van Tuyl</td>
<td>American Book Co., N.Y. 1935</td>
<td>None</td>
</tr>
<tr>
<td>Mathematics in Life Unit A Measurement</td>
<td>Schorling Clark</td>
<td>World Book Co., Yonkers on Hudson 1935</td>
<td>One sub-unit of seven pages on history, two large pictures of measurement through the ages</td>
</tr>
<tr>
<td>New Applied Mathematics</td>
<td>Lasley Mudd</td>
<td>Prentice Hall, Inc., 1935</td>
<td>Two articles, introduction to geometry and to algebra, two pictures</td>
</tr>
<tr>
<td>Practical Mathematics</td>
<td>Lennes</td>
<td>Macmillan Co., N.Y. 1936</td>
<td>Introductory paragraph</td>
</tr>
<tr>
<td>Child Life Mathematics Books 7 and 8</td>
<td>Overman Woody Breed</td>
<td>Lyons Carnahan Chicago 1936</td>
<td>Book 7: Four articles, one picture, Book 8: None</td>
</tr>
</tbody>
</table>
Evidences of the Successful Use of Historical Enrichment Material in the Junior High School Mathematics

As stated earlier in the study, we have no direct evidence of the effectiveness of any enrichment other than the experience of those who have given it a fair trial. Some educators have given such evidence through articles published in professional periodicals. Among these is Professor J. O. Hassler of the University of Oklahoma. The place of historical background in his philosophy of education in mathematics is shown by the following quotation from the preface of his recent book, *The Teaching of Secondary Mathematics*.

He says,

> The teacher should know his subject and the history of its development to such an extent that he not only can understand the important pedagogical relation of the more advanced topics to the part he teaches, but can win the admiration of his pupils for his ability and scholarship.

In 1929, Professor Hassler spoke before the Kansas Mathematics Teachers Association at Topeka on the subject, "The Use of Mathematical History in Teaching." His conclusions, based on his experience in the four grade high school apply equally well to either unit of the secondary school. The following quotations from this speech can be considered fair evidence

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of his successful use of historical enrichment.

In more recent years we have come to the realization that there is educational value in knowing the history of a subject; consequently we tell the pupil how and when the human race first discovered and proved ... important and useful parts of mathematics. ... A knowledge of the history of the development of the mathematical processes he is learning will kindle the pupil's interest in the subject matter. Any newspaper publisher knows that matters pertaining to people have a greater news value than matters pertaining to things. We pause in our reading because of the human interest, and the incident is indelibly stamped upon our memory. ... Why not just as true with subject matter to the pupil.

In the course of the article, he refers to many correlations of historical material with basic subject matter which in his experience had proven interest provoking and valuable. His final conclusions have a direct and significant bearing on the subject of the present study, in that they state the reaction of a well qualified person to the experiment of using historical enrichment of the mathematics curriculum in the secondary schools. We quote them as follows:

A knowledge of the history of mathematics on the part of the teacher gives him a source upon which he may draw to enrich and enliven his teaching.

A knowledge of the history of mathematics gives both pupil and teacher an appreciation of the subject and its inseparable and vital connection with the development of civilization.15


15 Ibid., pp. 167-168.
In an article entitled, "Mathematics Plus," by Florence Brook Miller of Cleveland, Ohio, which also appeared in 1929, the author related an incident from her experience with a ninth grade mathematics class which definitely denotes pupil interest in historical material. The following quotation is from her article.

Before clubs were generally organized in our school a request came from some of the members of an algebra class to form a club so that they might have an opportunity to learn more about some of the historical bits regarding the formation of number systems, origin of symbols, stories about early mathematicians and so on. It was not uncommon to have members of that class ask for some of the class time to give a mathematical puzzle or to explain a peculiar method of multiplying. Needless to say the club was formed and was a very lively and worth while one.

Lao G. Simons of Hunters College, New York City, made this statement in an article entitled "The Place of the History and Recreations of Mathematics in Teaching Algebra and Geometry."

There is no intent in this paper to make the history and recreations so prominent that the chief emphasis seems to be along that line, but they constitute one of the methods by which the work can be made interesting, significant, alive.

Another significant comment is found in a discussion of "The Social Qualities of Mathematics" by J. T. Rorer.


Wm. Penn High School, Philadelphia. It reads as follows:

History is one of our chief social studies. There is opportunity to teach much vital history in the mathematics classrooms. What part has counting played in the uplift of mankind? What economic part has algebra played? . . . What did Napier and Briggs do for us? The great Newton, Leibniz, Descartes and a host of other discoverers have done more for us than Julius Caesar and Napoleon Bonaparte. 18

The schools of Lakewood, Ohio, have been using an outlined unit on history of mathematics in their seventh grade course of study since 1932. In order to ascertain the results of this type of presentation, a letter was dispatched to Miss Grace Needham, chairman of the curriculum revision committee of Lakewood schools. Miss Needham, who is also the principal of Harrison School, responded graciously to the inquiry with the following reply:

As chairman of the committee that worked on the curriculum revision I am deeply interested in your reaction to history unit. Altho I am not entirely positive, I think we were the first to definitely include such a unit in the Junior High mathematics course.

The teachers who composed the mathematics committee have all majored in mathematics and are also well acquainted with the philosophy underlying the teaching of Junior High mathematics. The greater part of our first year's work was to study all material in the latter field.

The history unit was placed in the 7B grade because we felt it presented a new and interesting approach to the review of the fundamental operations with numbers. Both teachers and pupils are very

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enthusiastic about it and it opens up to the pupils the broader field of mathematics which is the work of the Junior High school.

The natural conclusion from Miss Needham's comment is that reasonable success has been achieved in Lakewood with historical content included in the course of study as a unit. There seems to be some doubt on the part of other educators as to the advisability of presenting such content in this manner. Arguments may be logically presented both for and against the unit plan but the probable solution is a compromise. Some phases of historical enrichment lend themselves to organization as a unit while others seem more logically presented as activities in connection with some other unit topic. Mr. Betz of Rochester favors the introduction of historical material as an integral part of the course and his letter indicates successful use of this plan. In answer to a letter concerning his textbooks and their use of history he replied as follows:

In reply to your letter of recent date, I may say that Junior Mathematics for Today is the textbook series used in all our Rochester schools. I think that I am also correct in asserting that this is the first series of textbooks in which historical backgrounds have been included as an integral part of the course. . . . In general, it is best to distribute historical materials throughout the course. At one time in the senior high school, I used to set aside two or three weeks each.

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19 Quoted from personal letter of February 10, '37, by permission of the author, Miss Grace Needham, principal of Harrison School, Lakewood, Ohio.

Quoted from personal letter of March 10, '37, by permission of the author, Mr. William Betz.
semester in the form of a separate unit. I do not think that this plan worked as well as the present plan of a more uniform emphasis throughout the entire course. . . . Naturally, the effectiveness of such teaching depends, in large measure, upon the enthusiasm with which these historical items are presented. An indifferent and slovenly teacher never gets results with any method.

The final comment of Mr. Betz places the responsibility for successful use of historical background in mathematics teaching where it must eventually rest. We have proof that it has been used successfully and is being used successfully, where fair trial has been given, regardless of which plan of presentation has been favored. The keynote of all the quoted comments seems to center on the well-informed and enthusiastic teacher. Could it be possible that junior high school teachers as a group are not well informed in the history of mathematics? It is not included as a required course for a mathematics license in the state of Indiana. Unless it has been introduced incidentally in connection with other courses by teacher training institutions there could easily be a lack of preparation on the part of many teachers to use such material as enrichment. Naturally such a teacher would not be enthusiastic over its inclusion in the course of study. The responsibility, then for successful use of mathematical history in teaching may extend beyond the teacher to the teacher training institutions and the course requirements for the mathematics teacher in the junior high school.

20 Quoted from personal letter of March 12, '37, by permission of the author, Mr. William Betz.
Summary

Karpinski, in an article which appeared in The Mathematics Teacher, says of mathematics, "In this study, the student must be made to feel himself the heir of all the ages." In this chapter we have endeavored to point out an educational trend at the present time toward the realization of this ideal through historical enrichment of the junior high school mathematics. Tabulations of recent courses of study and textbooks show an increasing use of such material since 1930. Evaluation of junior high school courses by educational authorities show its use in all courses listed as outstanding except two, and correspondence has established its actual use in the systems represented by these. Citations from published articles and personal letters from educators have given ample proof that historical material can be used successfully as enrichment of the junior high school curriculum if the teachers are prepared to teach it and appreciate its value. Since such preparation is necessary it has been suggested that the responsibility for successful use of such enrichment lies partially, in the requirements of the teacher training institutions.

CHAPTER III

HISTORICAL MATERIAL WHICH CORRELATES WITH THE
REGULAR JUNIOR HIGH SCHOOL MATHEMATICS

Such Correlation Exists and Can Be Used Effectively

Traditional ideas concerning the learning process have been consistently refuted by modern psychology. School subjects are no longer isolated behind artificial barriers with "No Trespassing," signs prominently displayed. No progressive teacher can fail to recognize the intertwining threads of interest which reach out from any phase of human endeavor and connect it with the past, present and future of the race. The geography, history, and civics, of the old curriculum have disappeared from the new, and in their place are the social studies, in which the same material is presented not as unrelated subjects, but linked together and inter-related as they actually are in the story of mankind. The social studies have in turn become the background for the study and interpretation of literature, music and art. Such social background belongs just as logically to the growth and development of mathematics. Geographical conditions have molded
its forms and expression; people of many races have contributed to its growth according to their peculiar racial characteristics; religion and social customs have stimulated or retarded its development; commerce and navigation have fostered its progress and carried its symbolism to the far corners of the earth. Philosophers and statesmen have sought mental recreation in unraveling its mysteries and thereby sharpened many a mathematical tool for the engineer of today. The mathematical research of one generation has led the way to new discoveries of science by the next. These historical facts are part of our cultural inheritance and they should be used as the natural link between mathematics and the social studies.

That historical material can be successfully used in this way has been proved by the writer's experience in the classroom. Monotony is almost unavoidable in the teaching of mathematics unless a diversity of material and new ways of presentation are definitely sought. Historical enrichment adds variety, and stimulates the curiosity of the pupils about the very skills they are practicing. Opportunities for its use may be given in the list of suggested activities for a unit. Stories and biographical sketches are effective when given as individual or group reports; maps and blackboard illustrations help to emphasize the contributions of different countries. Opportunities for dramatization can be found in the history of mathematics, and the making of pictures or
models of the abacus never fails to delight some ambitious craftsman. Demonstrations of reckoning with the counting boards or a display of medieval finger reckoning, however crudely they may be performed, arouse the interest of every member of the class. Trying to calculate with ancient number systems, solving old puzzle problems or making magic squares intrigue some pupils. The magic stunts which can be performed by the "casting out nines" method of checking not only affords some amusement but imparts some very worth while information to the young accountant. Stories of the old man measurements can be told effectively, and the display of an old arithmetic of the early nineteenth century with its strange units of measure and queer rules will add interest to the stories. Many other activities will suggest themselves to the alert and well informed teacher as occasions for them arise.

From the experience of authorities quoted in Chapter Two historical material has been proved successful in the introduction or overview of a new unit. Sometimes a subject such as taxation, for example, has been discussed in the home until the junior high school pupil thinks he knows all there is to be learned about it. Such a subject gains new significance through a presentation of its age old influence on world history. New interest in a review of the fundamental processes has been aroused by the story and demonstration of other
number systems, tried and discarded by our ancestors in favor of the better one we now use. By experience, the writer has found that the stories of Egypt and Greece and the conditions which fostered the study of geometry in these two countries have made an effective introduction to intuitive geometry. Such correlations of history with the immediate mathematical skill to be taught relieves monotony for both pupil and teacher, develops pupil ability in individual investigation, broadens the view of the subject beyond the mere manipulation of figures, and develops new respect for processes which represent human achievement so valuable as to outlive the centuries. Personal experience as well as the testimony of Miss Needham and other teachers cited in the preceding chapter has indeed, led to the belief that such background gives to those who have mastered mathematical skills, a better understanding and appreciation of our number system and to those less skilled, a new stimulus to their interest and desire to acquire proficiency.

Meager Sources of Reference Material for Pupil Use

When attempts are made to find reference material to be used in such activities as have just been suggested, its meagerness is quickly noted. Many excellent and comprehensive histories of mathematics are found in adult libraries but very few in the children's departments. There is little readable material except a few short and scattered encyclopedia paragraphs which can be placed in the hands of junior
high school pupils who feel the urge to read and investigate a subject for themselves. This conclusion was reached after a careful examination of library files and the bibliographies offered by Columbia University and the National Educational Association, Research Division. A careful investigation of all suggested references was made and the results summarized in the list which follows.

The only book on the subject actually written for the child reader is a little volume called _Number Stories of Long Ago_ by David Eugene Smith of Columbia Teachers College. It is limited in content to the story of the numerals and number system and told in the style of the bedtime story. While an attractive little book it is better suited to the elementary grade level than to the junior high school. The Committee on Materials of Instruction of the American Council on Education has issued a set of pamphlets entitled _Achievements of Civilization._ The second of this series, a thirty-two page booklet called _The Story of Numbers_ is very well prepared and is an excellent reference for the junior high school pupil. The subject matter is limited to the early numerals and number systems, the use of the abacus in its various forms and a
brief mention of the development of the processes in written calculation. Numbers Three and Four of this series entitled, The Story of Weights and Measures and The Story of Our Calendar\(^3\) are also readable and helpful but limited in content. During the Century of Progress Exposition at Chicago, the Colortext Publications published a colorful little twelve page paper pamphlet entitled Numbers through the Ages and around the World\(^4\) for which Louis Karpinski wrote the brief text. This pamphlet was one of a series called A Century of Progress Wonder Library, and was distributed on the exposition grounds. For brief references and attractive pictures this booklet can be used effectively, but the subject matter is naturally limited to early numerals, the abacus and finger reckoning. In Compton's Pictured Encyclopedia\(^5\) and The World Book\(^6\) are a few good historical paragraphs under various headings such as arithmetic, algebra, geometry, abacus, numerals, mathematics and the names of well known mathematicians. Aside from these references, no material for child readers was found.

Books which may prove very helpful to the teacher are

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The History of Arithmetic by Karpinski, A Short History of Mathematics by Vera Sanford, The Story of Reckoning in the Middle Ages by Florence Yeldham, and The History of Mathematics in two volumes by David Eugene Smith. These books are especially interesting, readable and comprehensive for an adult student, but will give little help to the average child of junior high school age. It seems reasonable, however, to believe that historical material could be gathered from various sources and arranged to supplement the junior high school course of study, without including that which concerns only the higher mathematical studies. No such work which can be called adequate or complete seems to have been done. Upon the theory that there is both a need and a possibility for such a compilation, the second part of this study was attempted.

Basis for Choice of Correlating Material Which Should be Found in Such a Compilation

The basis for choice of such material should be the experience of the average pupil who might be expected to make use


of it. No material should be introduced which does not concern the facts and skills of mathematics with which the junior high school pupil is familiar or will study during his course. In order to proceed logically to such a choice of historical material it is necessary to know just what are these facts and skills with which the pupil should be familiar at the junior high school level. It is a recognized fact that children are interested in experiences in which they themselves have shared and seldom interested in the accumulation of knowledge which cannot be connected in some way with these experiences. The fact that their ten fingers suggested to men the number system now used all over the world, should, quite naturally, be of more interest to a seventh grade child than the knowledge that Descartes was the first man to write an analytical geometry. In order, to select relevant historical facts which might interest the child it is necessary therefore to determine what subject matter he has actually had.

From the survey of courses of study discussed in the preceding chapter and a glance through several elementary textbooks it was found that the seventh grade pupil enters junior high school with certain basic skills in mathematics fairly well mastered. This fundamental knowledge might be summarized as follows:

A seventh grade pupil should know something about:

Number words used in counting and the Hindu-Arabic
numerals which correspond to them

The common mathematical symbols such as $+,$ $-,$ $\times,$ $\div,$ $=,$ and $0$

The decimal system of writing whole numbers and fractions

Roman numerals at least to one hundred

The four processes of addition, subtraction, multiplication and division with whole numbers and fractions

The form and use of the common fraction

The United States money units

The ordinary units of weight and measurement

The meaning of the per cent fraction

During his junior high school course a review of these fundamental skills will continue and other subject matter will be introduced such as:

Applications of percentage to business practices

Banking practices and investments

Household problems and measurement

Graphs and statistics

Taxation and insurance

Proportion and indirect measurement

Intuitive geometry

Elementary algebra with stress on the formula

Any historical allusions pertaining to the subjects listed, may be regarded as legitimate and educationally sound correlation in the teaching of mathematics to junior high school pupils.
Summary

In this chapter, the theory that historical material correlates with mathematics has been discussed. The adherence of this theory to modern ideas concerning the curriculum and the listing of many possible activities in which it has been used seem to give reasonable proof that mathematics needs the historical background to vitalize its content. The variety of ways in which it can be introduced has been mentioned and pupil activity stressed. There are, however, few books on the history of mathematics which can be consulted profitably by the pupils themselves. These have been named, together with a few excellent adult references. The suggestion followed, that a compilation suitable for the junior high school grades could and should be made, with the material chosen on the basis of relevancy to the subject matter already taught in mathematics.

The hypothesis upon which the study is to proceed from this point might be briefly stated in this manner:—In the choice of historical material suitable for inclusion in a junior history of mathematics, only such material should be considered as can be directly or indirectly related to the mathematical facts and skills already possessed by the pupil. Such material should also be linked with the child's immediate experiences, whenever possible, through his general knowledge of social and geographical background. Finally,
the diction of such a compilation should not be complicated beyond the young student's ability to read and understand it easily.

The chapters which follow are designed to meet the conditions of such subject matter but, due to the limitations of such a dissertation as this, no effort has been made to introduce direct appeal to the child through illustrations and personal questions. If such material were actually being prepared for the press both illustrations and suggested activities should occupy an important position throughout the entire text.

The modern child learns to count and use our number system which has made possible the inventions and discoveries of science accepted as a part of our heritage without comment, as quoted by Cajori, has said, "I am sure that no subject loses more than mathematics by an attempt to dissociate it from its history."1

The comfort and ease with which complicated tasks may be performed, we owe largely to the labors of our fore­

PART TWO. THE COMPILATION OF HISTORICAL MATERIAL WHICH CORRELATES WITH THE J. H. S. MATHEMATICS COURSE

CHAPTER IV

THE STORY OF NUMBERS

When the modern child learns to count and use our number system he casually accepts that which has cost the human race untold centuries of effort to develop. He is taught to count long before he exists as a separate being, and he learns that the history of modern inventions but the number system which has made possible the inventions and discoveries of science is accepted as a part of our heritage without comment.

Glaisher, as quoted by Cajori, has said, "I am sure that no subject loses more than mathematics by an attempt to dissociate it from its history."

The comfort and ease with which complicated tasks may now be performed, we owe largely to the labors of our forefathers. Life, however complex it may seem, is infinitely more easy to live than it was in the days of the cave man.

The making of tools for physical use was one of the first
indications of man's superiority over other forms of animal life, and his first tools for mental use, such as number symbols and forms of writing, hastened his advance toward civilization immeasurably. The school boy of today begins his mathematical study with a set of tools almost perfect in design, polished and sharpened by the trials and experiments of mankind through the ages. Knowledge of this long past should lead boys and girls to a greater appreciation and regard for the study which has given to the world the possibility of modern science and invention.

Mathematical law and proportion existed in the natural world long before the existence of human life. Man did not invent these relationships, but he has through his long upward climb, found many ways and means of discovering, comparing and expressing them. This knowledge is that which we call mathematics or the science of exact relationships. According to the opinion of Keyser,\(^2\) mathematics begins in the ability to discriminate multiplicity as in the recognition of two objects instead of one, and is therefore, the most primitive achievement of mind. Many lower animals seem to share with primitive man the ability to count or discriminate multiplicity in small groups. In litters of four or five young ones the parent will

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note the absence of one and hunt for it. Birds seem to know when the correct number of eggs are in the nest. Sheep dogs are said to know when one of the flock is missing. Animals also seem to recognize efficiency in the use of geometric forms as, for example, in the hexagonal cells of the honeycomb, the symmetry of many birds' nests and the regular bracing of the spider's web. Another interesting observation of mathematical sense in animals is the path of wild animals in crossing a gully. They invariably make use of the angular or zig-zag path which civil engineers have found most effective in building mountain roads. Sense of direction and distance are well-known animal instincts, and primitive man must have begun with mathematical concepts not far in advance of the animals. But with man, counting was only the beginning and not the end of his abilities. Counting undoubtedly developed in the infancy of the race and its contemplation naturally leads to curiosity concerning the first number words.

It would be most interesting to turn back the clock of time to those prehistoric years and really find out what were the first number words, but this is quite impossible. We can, however, find the next best source of information in the vocabularies of primitive peoples still living in isolated parts of the earth. Such investigations have been made among tribes living in remote regions of Australia, Africa, South America, and the Pacific islands. Professor Levi Leonard Conant has
told many interesting stories of this research in his book, The Number Concept. He says that the number words of primitive tribes are entirely concrete or associated with certain objects which they used in counting. When we use the word three, it carries a meaning without being connected with any particular group of objects, but the word used for three by the savage might mean three pebbles, three fingers, three grains of wheat, three sticks or shells, according to what had best served his purpose as concrete counters. One tribe of island people have number words which mean one breadfruit, two breadfruit and three breadfruit because that tropical fruit happens to be a very important part of their diet.

Such words are the exceptions, however, and not the rule for most primitive number words indicate that the fingers have been used for counters. The word for one has been found to be the same as the word for finger in languages unrelated in any other manner. A very interesting survival of this association is found in our own language for we use the word, digit, to mean any one of our nine numerals while digit also means one of the fingers or toes.

Primitive life does not demand a complicated set of number words and among a few very backward tribes of Brazil and parts of Australia, counting still does not extend beyond two or three. Several tribes have words to count as far as seven but very few as far as twenty. Native Eskimos are said to be

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unable to count farther than fifteen and the Tasmanian native language had only words for one, two, and plenty. Shubert, in one of his mathematical essays, says that African tribes sometimes pass on to a neighboring tribe the number of enemies that are approaching by placing at a certain spot a number of stones exactly equal to the number of invaders. Thus the number idea is conveyed without the use of a number word at all.

Schubert also expresses the opinion that the very first number words men ever used were natural number words or the repetition of the same sound. An example of such words for the first four numerals would be something like this: oh-, --oh-oh-, --oh-oh-oh-, --oh-oh-oh. No such number words have actually been found in any native language existing today and if they once did exist the time is so far beyond history that the number words connected with counters have long since replaced them. However, we listen to such number words expressed mechanically by the striking of the clock, and they are also the basis for the numerals of the continental code in radio.

The fingers were doubtless used in sign language for numbers long before words were used. This must have been true because such sign language was common among traders of different races during the Middle Ages and indications of it

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In fact, a very elaborate finger language is used on the floor of grain markets in our large cities by the traders. But the best proof lies in the actual number words still in use among the primitive tribes which refer to the fingers and hands. The word for five is frequently the same as the word for hand while ten is expressed by a word meaning two hands. For fifteen, languages of tropical tribes nearly always show a word meaning one foot while twenty is called a man. This shows that the toes as well as the fingers were used as counters where they were ordinarily exposed. Different number words would actually be used. So must count.

All primitive counting is done by what is called the additive system. For example, an Australian bushman might have number words for one, two, three, and four which indicate his fingers. He would then have a hand for five, hand and one for six, hand and two for seven, and so on up to hand and five for ten. This would probably express his arithmetic, which he would need in his simple life.

Large numbers were hardly ever needed. But such a meager vocabulary for numbers would be complicated by different languages which did try to make combinations of number words. They would try to make combinations of number words and devise, probably, began in the early Arabic period. We know very little except that in the Middle Ages they have come to us through Levi Leonard Conant, chiefly from the Latin
cites an example from the language of the Api, a New Hebrides tribe, which is quite interesting. This tribe evidently did not use the toes in counting for their word for ten meant one man. For two hundred, which was beyond the expression of most primitive people, they said, "duulimo toromo mo va juo." In literal translation, this word actually said, "Ten times the whole man taken two times."

Without serious thought, most of us would probably say that we use many number words, but this is not really true. Suppose it were possible to count to a million; let us see how many different number words would actually be used. We must count all the words from one to twelve for they are different, but thirteen really says three and ten and they have already been included. Twenty seems to be different but it is really only a contraction of two tens and cannot be called a new word. All of the compound words are repetitions of old words. One hundred is new, and not another new word is used until we reach one thousand. After one thousand no new word is added till one million is reached. By that time our voices would doubtless be worn out yet we would have used only fifteen different number words. That is only one of the many advantages which our number system has over any other system ever devised.

Of the origin of our own number words we know very little except that in their present form they have come to us through various European languages, chiefly from the Latin
and the Greek. If they once had significance in the counting of objects, as the words of savage tribes still show, it has long since disappeared. We do know that by the use of our compound words there is no limit to the numbers which can be verbally expressed. But that does not necessarily mean that our conception of numbers is infinitely great. How many of us can look over a large crowd and accurately estimate the number of individuals in it? Do you know how long it would take to count a billion dollars if you counted a dollar a second, day and night without stopping? It would take over thirty years at that impossible rate. Why do we use the word million so carelessly in exaggerations, unless the word means little more to us than the Tasmanians' word for plenty? Yet modern life contains many situations which demand the use of large numbers and we are most fortunate in having a number system which is adequate to express them both verbally and in written symbols. Written numbers have an even more interesting history than number words, and this will be set forth in detail in the following chapter.
Spoken language has always been a constantly changing medium of expression, but written records have been preserved just as they were when primitive man first scratched them on rock and clay. They long awaited the archaeologist's spade and the modern scholar's research to tell their story; a thrilling story of man's crude and laborious efforts to pass on to others, his hard-earned knowledge. Because of such records, we have some idea of how man first attempted to express number concepts in written language.

Prehistoric men who lived in the caves of Europe may have scratched or painted symbols on their walls which meant number to them. We are not sure of this interpretation, but the repetition of pictured objects was not unlike the pictured numbers of the American Indian. The Indian conveyed the idea of three canoes by actually drawing pictures of three canoes. Five enemy warriors on the trail were shown by a row of five men, distinguished as enemies by some familiar tribal symbol. Such picture numbers were in common use by the Red men when Columbus discovered America. The Egyptians, with their
extremely ancient civilization, had reached the picture number stage of development perhaps eight to ten thousand years before the American Indian. Asiatic tribes have also left inscriptions which prove their early use of such numbers. The method of expressing number by concrete picture forms is probably the most primitive of all notations and it is logical to believe that the next forms, called natural numbers, developed from it. Natural numbers are those which convey their meaning by the repetition of some symbol less difficult to make than the concrete picture. Parallel lines or scratches, either vertical or horizontal in position, have been the most common symbols used in natural numbers. Small circles or dots, wedge-shaped marks, notches in sticks and knots in strings are other well known symbols. Survivals of these ancient number forms are found among numerals still in use such as the Roman numerals seen on the clock face and in chapter headings. Natural numbers are found in the forms I, II, and III and also in the repetition of X, C, and M for larger numbers. The Chinese use natural numbers for one, two and three with the repeated lines in the horizontal position like those of a musical staff. We still use this system of writing numbers when counting votes or scores by means of marks like these, IIII, and we certainly see and read natural numbers each time we play games with the dice, dominoes, or cards.
Among ancient races, the Sumerians or Babylonians, and the Egyptians have left the most complete record of number symbols used in notation. Before the era of written history, each of these races had achieved a civilization far beyond the primitive stage. Each had developed a form of writing and a number notation as early as 3000 to 4000 B.C. The notation in each case had advanced beyond the picture number stage.

The Egyptians found a medium for permanent records in the rock walls of their monuments and tombs. Here they left written numbers which reveal the use of a decimal system and a remarkable ability to express large and complicated numbers. They used the natural repetition of a short vertical line to express the first nine numerals. A new symbol, somewhat like an inverted U, was used for ten, and repeated for multiples of ten. The sign used for one hundred resembles our bass clef sign in music. The symbols for one thousand, ten thousand and one hundred thousand are supposed to resemble respectively, a lotus flower, a pointing finger, and a polywog. All these symbols were hieroglyphic in form because they were modifications of some earlier symbol with concrete meaning. For example, the symbol for a million was a crude representation of a man with his arms raised above his head in astonishment at so great a number. All these symbols were repeated for multiples of the original value and no position value was used. Much later, this notation was simplified to a system known as
Among the relics of an old civilization discovered on the island of Cyprus, have been found stone tablets on which natural numbers have been carved. These numbers, however, are quite different from those used by the Egyptians. They most nearly resemble symbols used by the ancient Cretans who were a people of pre-Greek culture closely related to the Trojans. The Phoenicians are believed to have been the remnant of this race which survived the Greek invasion, for they too used a similar number notation. Thus, these old number lines have helped to establish the common origin of these early sea-faring peoples, even though they have not yet been entirely deciphered.

The Sumerians who lived in the fertile valley between the Tigris and Euphrates rivers, had no quarries from which to obtain building stone. Consequently, they had to depend upon clay for building material. The sun-dried brick may have suggested to them the use of clay tablets for writing and preserving records. The dry climate helped to keep many of these old clay tablets intact and during the past century they have been found and deciphered. To write in the clay, these people made use of a stylus or blunt pencil and the peculiar little imprint made by this pencil was triangular or wedge-shaped. For this reason, their writing and their
number symbols have been called cuneiform or wedge writing. They had developed a very unusual number system with two distinct bases, the common decimal base and the base of sixty. However, they still used natural numbers for the first nine numerals. These were made by repetition of the wedge mark with the point down. Other positions and groupings of the little wedge marks were used to indicate larger numbers. New symbols were used for 60, 3600 and other powers of sixty while in some cases different symbols were used for powers of ten. The numbers based on sixty, were used for their astronomical records which were very complete, and they show the first recorded use of place value in any notation system. The same symbol was used for sixty as for one, with a character which evidently denoted the absence of number or zero.

Strange to relate, no European people ever seemed to grasp this idea of writing numbers until after the Hindu-Arabic of transactions by the tally system. After the system numerals were introduced thousands of years later.

The oldest Chinese numerals known to us today are called hunger. They were usually over-related with a live rod numerals because they resemble the short bamboo rods which were originally used for computation. They used both the horizontal and vertical position of the rods and many other combinations which were reproduced in written numerals. The Ancient Greeks used a repetition of eight circles for Japanese and Korean numbers were very similar to the rod numbers, and were probably modifications of the older Chinese forms. The Hindus seem to have been an exception in the use
of natural numbers. However, little is known of India's early history and their use may have preceded any known records of written numerals.

Another rod system was in vogue in England during the Middle Ages, but was not used for actual written records. This was the tally or notched stick system of accounts. A debtor recorded his debt by cutting notches in a stick. The stick was split and both borrower and lender received a piece. When the latter produced the notched stick which matched the piece held by the debtor, it was said to tally and prove the amount correct. Banks also used the tally system. Depositors held notched sticks or stocks to match those held by the bank and this is said to be the origin of our modern word, "stockholder." An old story is told of these tally sticks which indicates their extensive use.

Up to 1543, the British Government kept records of transactions by the tally system. After the system ceased, the basement of the House of Commons remained cluttered with vast accumulations of these dry sticks for nearly two centuries. Finally it was decided to burn them. The stove became over-heated and a fire ensued which burned down both the House of Commons and the adjacent House of Lords.

In the western hemisphere, many interesting notations which represent the use of natural numbers have been found. The Aztecs of Mexico used a repetition of small circles for

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numbers up to nineteen and a change of symbol for twenty. This system of counting by twenties was common among native races of America. The Peruvians of South America had a very ingenious method of recording numbers without writing them. They tied knots in strings of various colors. Small knots were repeated for numbers less than ten and large knots were used for tens. This set of cords was called a quipu and Karpinski suggests that it may have been their method of recording the census in various districts.

The Mayas of Yucatan whose civilization so strangely disappeared before the coming of the white man, had developed one of the most complete number notations found in the new world. That they were also familiar with many astronomical facts is proven by the inscriptions on their hieroglyphic monuments and date stones. Their calendar was only slightly less accurate than the one we use today and their number system had many of the features of our present system. They had a symbol for zero that resembled a shell or oval disk and they used a position system on twenty for writing large numbers. When changing the value of a figure, they moved it one place above the former position rather than one place to the left, as we do in our system. Their numerals were written with small circles and horizontal lines, a circle meaning one

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and a line, five. These were used in natural number form for the numerals from one to nineteen. Thus, seventeen was written with three lines and two circles, but twenty was one circle raised to the second position with a zero sign beneath it. This corresponds to our method of writing ten in figures. Except for the greater difficulty of writing the natural numbers the Mayan system compares favorably with the Hindu-Arabic system now in use and is decidedly less cumbersome than the Roman system used by Europeans for more than a thousand years.

The many illustrations of number writing among early peoples seem to point to this conclusion, namely, that the writing of natural numbers was one of the few examples of parallelism in the development of the human race. The same ideas and frequently the same symbols for expressing them were developed by widely separated people all over the world, with little or no opportunity for diffusion of this knowledge. Thus, natural number writing may be compared to finger counting in its almost universal and independent development.

The use of natural numbers among people of primitive culture was gradually modified and changed as life situations became more complex. An interesting variation in symbols was the extensive use of letters to express number value. Sometimes these letters were the initial letters of number words but often they were assigned value according to their order in the alphabet. The Greeks developed the latter method most
extensively but at best it was a cumbersome notation, poorly adapted to any calculation. This letter system probably accounts for the fact that the brilliant Greek mathematicians did very little to advance the study of arithmetic while geometry, which does not depend to any great extent upon a good notation, reached its height in their schools.

The Hebrews were another race that practiced the writing of numbers by means of letters in alphabetic order. An interesting side light on Hebrew notation is given by Smith\(^3\) in his historical sketches. It seems that their system gave rise to an ancient numerology fad called "gematria." It resembled some of the modern forms of fortune telling in that it was based on the numerical value of letters used in the spelling of names. The Bible has one reference to this strange mysticism which has caused much conjecture. In the book of Revelation, the beast is said to bear the number, 666. This was probably written as a veiled condemnation of Nero, for in Hebrew, his name gave a total number value of 666. Radical church leaders, however, have assigned this number to some religious or political enemy of practically every century since the book of Revelation was written. Another example which indicates the common use of gematria among Christians of the first few centuries is the use of the number 99 in

the closing of prayers recorded in early manuscripts. The word, "amen," in Hebrew totalled 99 in number value and the number appears quite casually as a substitute for the word.

The Roman notation followed the Greek idea in the use of letters but was much less involved. Some of the letters such as C for one hundred and M for one thousand were undoubtedly initial letters of number words but the original significance of the others is not definitely known. The Romans used natural repetition and comparatively few letters for the first ten numerals and, thereafter, for multiples of ten and one hundred. The Greeks combined their letter numerals by an additive method with no position value. The Romans, in their later number forms, introduced a few examples of subtraction such as IV for four, IX for nine, XL for forty and XC for ninety. This was a departure from the general additive method in compound numbers which had prevailed up to that time and possibly prepared the way for more drastic breaks from old methods in the multiplication-place value system soon to be brought into Europe by the Arabs. Poor as the Roman numeral system was, it was the best of its time, and Roman conquest helped to introduce it throughout all Europe. These numerals were actually used by Europeans for about fifteen hundred years and we still use them in our own country for many purposes.

Early number systems were all grouped around certain
bases from which compound numbers were formed. If letters were used as symbols they stood for multiples of this base and if repetition of marks was the method of writing numbers a different mark was used for the base and its subsequent powers. When counting was confined to the fingers of one hand, the base, two, was probably used and one, two, two and one, and two and two constituted a binary number system. Such a system could be written with but one repeated symbol and was extremely primitive. The prevalence of such words as pair, couple, and brace in our language may indicate the existence of this old binary system in the dim past.

The systems actually in use since the dawn of history have used the three bases, five, ten, and twenty with only a few exceptions. Of these three, the decimal or "ten," system was the system which finally prevailed over the world. The following paragraph from Schubert concerning number words applies as well to written numerals.

The languages of both civilized and uncivilized peoples always construct their words for larger numbers from words for smaller numbers. What number we shall begin with in the formation of compound numeral words is quite indifferent, so far as the idea of number itself is concerned. Yet we find, nevertheless, in nearly all languages one and the same number taken as the first station in the formation of compound numeral words and this number is ten. Chinese and Latins, Finns and Malays -- that is -- peoples who have no linguistic relationship, all display in the formation of numeral words the similarity of beginning with the number ten the formation of compound numerals. No other reason can
be found for this striking agreement than the fact that all the forefathers of these nations possessed ten fingers.  

The earliest of the three common systems was probably the quinary or "five" system. Several African tribes are using a quinary system at the present time but only traces of it are found in the written numerals of more civilized races. In such a system, a different mark or symbol is used for five, and six is made by a combination of the symbols for five and one. The Roman numerals, while based on ten, have the evidence of this earlier system in the use of a new symbol for five and again for fifty and five hundred.

At some far distant time, our European ancestors used a vigesimal or "twenty" system which has left its trace in the languages of the present day. The English word, score, meaning twenty is commonly used for a number word while the French say quatre-vingts or four twenties for eighty. But the early peoples of America consistently favored twenty as a number base. The Mayan system mentioned earlier in the chapter shows a decided trace of the quinary base in the new symbol which is used for five and then repeated with the unit symbol in natural form for numbers up to twenty. The additive method was used in these first nineteen numerals but for the base twenty the Mayans placed a position value on the numeral for one and used a symbol for zero to indicate this position.

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The gradual introduction of place value and the use of zero established more firmly the decimal system which was to replace all other systems. The old systems are of interest to us today, only from the standpoint of history and the traces they have left in our language or customs. Two of these old systems which were abandoned long ago have had decided influence on our every-day life. The more unusual of the two was the sexagesimal system based on sixty which was in use by the ancient Sumerians more than three thousand years before Christ. These people who preceded the Babylonians in the Tigris-Euphrates valley were interested in astronomy and the reckoning of time. From their observations, they believed the year to consist of three hundred sixty days and they originated the idea of measuring the circle by the same number of divisions. Whatever may have first suggested the use of sixty as a number base, a system was very consistently carried out in cuneiform symbols on this base. There is a distinct heritage from this old system in our reckoning of time by sixty minutes and sixty seconds as well as in circle and angle measurement.

The other old base which has left its mark in the language and usage of many races, including our own, is the duodecimal or "twelve" system. Some people consider this base much more convenient than our decimal base and favor its adoption, but since men were not born with twelve fingers, the decimal system will doubtless remain as it is. Our word dozen is a constant reminder of its former use, however, and possibly
the words eleven and twelve which are not true compound words, may suggest its use. The Romans used duodecimal fractions in measurements and our words, "ounce," and "inch," both owe their origin to the Latin word, "uncia," meaning the twelfth part -- the pound originally consisting of twelve ounces.

Thus far, we have scarcely mentioned the most important number symbols in existence, the Hindu-Arabic numerals now in use throughout the modern world. Because of their importance in the history and development of the human race we have given them the final consideration. They have been commonly called the Arabic numerals but modern investigation has proved that they are wrongfully so called and should be termed the Hindu-Arabic numerals. The Hindus were an oriental people given to philosophic thought and mysticism and number appealed to their intellectual curiosity. It is not strange that they should have originated the best system of writing numbers that the human race has invented.

Much of their early history has been lost but the few primitive writings that have survived are full of strange and mystical problems, written in poetic form and very difficult to interpret. They reveal, however, an ability on the part of the Hindu priests and scholars to read and write extremely large numbers by the use of nine peculiar number symbols. What these symbols originally were, nobody knows. They may have been priestly signs with religious significance or they might have been initial letters of some forgotten number words. The
important thing is not so much their origin as how they were being used at the dawn of history. The earliest trace of these nine peculiar figures which lack the sign repetition of natural numbers, is said to be in a rock-carved inscription found in Central India. This inscription may have been made as early as the second or third century before Christ and at that time no zero symbol had been invented. Nevertheless, during the centuries when Greece was struggling with her alphabet numerals and Rome building up her cumbersome letter system, the Hindu priests had begun to use these nine symbols with place value, based on the ratio, ten. Can you realize the importance of this fact to the modern world? How much progress could have been made in science and mechanical invention with nothing better than the Roman numerals for a number system?

For many centuries these symbols seem to have been used without any symbol for an unoccupied number position. It seems impossible to us, but since the Hindus considered mathematics from a theoretical rather than a practical point of view they were able to evade the use of a symbol by writing or reading each position by name rather than by number families. In this way words served the purpose of declaring the position. They would have read the number 204,025 thus: two hundred thousands, no ten thousand, four thousands, no hundred, two tens and five. For computing, they may have resorted to the counting board, but there is less evidence of its use in India than in any other country.
Some one more ingenious or perhaps more lazy than his fellows must have decided that a little 0 would be much more convenient for reserving a number position than written words. Possibly several people experimented with its use before it became generally popular. We only know that sometime before 875 A.D. it had been in common use for the zero appears in a Hindu manuscript of that date. It also appeared in the manuscripts of the Arabs about the same time, but they indicated it by a dot rather than a circle.

Without this zero symbol, the nine numerals would probably have found a historical grave in the land of their origin as did many other primitive number systems, but with it, the possibilities of this new system were unlimited. The Arabs who were the people of greatest culture during the early middle ages, recognized this fact and adopted the numerals for their own use in trade. Arab caravans traveled to far-off India and China, and brought the rich fabrics and jewels of the Orient to the Mediterranean cities. Along with these things they brought the learning of the Orient as well. They used the strange Hindu figures instead of the Roman letters and scholars in Spain and Italy became interested in the new system. About 1202 A. D. Leonardo of Pisa, also called Fibonacci, who was the greatest mathematician of his day, wrote a book about this new method of writing numbers. He gave credit for it to the Arabs and named the zero symbol,
"zephirum," after an Arabic word. This word was the origin of our word, zero, and also the word, cipher. The numerals are to this day commonly called the Arabic numerals because the Arabs advertised them to the world.

Leonardo wrote his book over two hundred years before printing was invented. No standards were fixed and no one knew exactly how to make the new Hindu symbols. Naturally there was great variation in their form. If you should try to read the numbers written by Leonardo, you probably would see little resemblance to our modern figures. In fact most of them would seem decidedly upside down or backwards. Only the one and the eight have kept practically the same form. During the centuries, while these symbols were going through many changes their meaning and use remained the same and now that the form has been definitely fixed through the invention of printing and calculating machines, they may be seen all over the civilized world.

Nevertheless they were not adopted as readily as one would imagine, even after their superiority had been demonstrated. Textbooks taught both systems, the Hindu and the Roman, side by side and also included instructions for using the counting board as good measure. Roman numerals, Hindu-Arabic numerals, and number words might all be used in the writing of one large number. People of medieval Europe were so prejudiced against change of any kind that they declared the new figures were devices of heathen magic and dangerous
for Christians to use. Miss Sandford makes this statement:

The recognition of the value of the symbols was so slow that nearly a century later than 1202, Florentine bankers were forbidden to use them and booksellers were obliged by law to mark their stock, 'not in ciphers, (Hindu numerals) but in intelligible letters.' (Roman numerals).\footnote{Vera Sandford, A Short History of Mathematics. Boston: Houghton-Mifflin Co., 1930, Chap. II, p. 95.}

After the invention of printing however, the use of the new system gradually replaced the old Roman system and the counting boards.

This decimal system with its nine numerals and a zero, based on the position value of its written symbols seems so plausible and necessary to us that we can scarcely imagine a time when men used number at all without its aid. The transition from the first crude forms of numbers to this most concise and complete system man has been able to devise, has come slowly. Trial and failure have been as frequent as trial and success. Long periods of time have elapsed during which scarcely any progress was evident; other periods, comparatively short, have produced far-reaching and important results. The perfection of a number notation to meet the complex needs of modern life has finally been realized in the Hindu-Arabic decimal system, but it is a cumulative heritage from the efforts of mankind since the first crude numbers were scratched on the rocks.
CHAPTER VI
NUMBER THEORIES AND OLD METHODS OF CALCULATIONS

Arithmetic, as we think of it today, did not exist in the ancient world. Scholars in Babylonia, India, China, and ancient Greece studied numbers in order to learn their interesting properties or their mystic powers; to discover hidden meanings or to develop unusual combinations and series. Arithmetic, and calculation necessary to common trade were two different things. Only mature scholars studied the former; the sons of the merchant class and common artisans studied the latter. The word arithmetic is a Greek word meaning number and was used to indicate the theory of numbers. Common calculation was called logistic. These distinctions were carried down through the centuries until the beginning of the modern era. As the merchant's position in society became more honorable, and education more general, the name, arithmetic, gradually assumed its present meaning of calculation as applied to business practices and simple measurements. With its ancient meaning, arithmetic was not a subject for boys and girls to attempt in school, but arithmetic as we know it today can be mastered in a few years. Let us find out what...
Primitive people and early civilizations developed many strange superstitions and mystical beliefs about numbers. Some were linked with religious rites, others with theories about the origin and nature of the universe. The number, seven was considered especially lucky or favorable with the gods. Odd numbers meant good luck, while even numbers brought bad luck. The Chinese called even numbers feminine or earthly, while odd numbers were masculine or divine. The Hebrews considered seven the number of completeness or entirety, as indicated by the Bible in the "seven churches," "the seven candlesticks," "the seven devils," "the seven fat and lean cattle" and the forgiveness "seven" times under the Mosaic law. The Aztecs of Mexico thought that seven was the number of the universe. Their explanation was quite naive -- there were six directions, east, west, north, south, up, and down and the spot where the observer stood made seven and completed the universe.

Four was also a revered number among the Aztecs and many of their myths reveal this fact. They said the rain god poured water upon the earth from four great jars. One contained good water, one, bad water, one, water that congealed, and the fourth, water that yielded no fruit. This water came from the four corners of the earth, but only the east rain was good.¹

Priests of India and China taught that numbers explained everything in the natural world. Pythagoras, the Greek scholar studied in the East and taught number mysticism to his secret order of mathematicians. He believed that the earth, fire, water, air and the heavenly spheres originated in geometric forms and that physical characteristics such as color, health, cold and heat were caused by number. Perhaps he sensed the future revelations of science concerning vibrations and wave lengths. At least, he believed mathematics to be the great conception of wisdom and its four divisions were arithmetic, music, geometry, and astronomy.

These are but few of the many number superstitions which could be cited, and there are interesting reminders of them in our common speech. Karpinski says,

The terminology of present-day arithmetic and some current phrases bear evidence of the continued influence of the mystic element in numbers. Such expressions as, "luck in odd numbers," "lucky seven," and "all good things are three," carry us back in spirit and even in content to the mysticism of numbers as practiced first in Babylon and then in Greece and Rome.\(^2\)

Odd and even numbers were recognized by the Egyptians and all the early peoples of the Orient. Prime numbers were also known and methods for finding them, worked out. The Babylonian priests taught astrology and connected their favorite number, sixty, with its mysticism. While the Greeks were less

superstitious than other ancient races they loved to search out strange properties of numbers. They considered numbers, geometrically. Prime numbers they called linear numbers; numbers of two factors were plane numbers and those with three factors were solid numbers. Plane and solid numbers were divided into figurate numbers, such as square, triangular, oblong and pentagonal numbers. Square numbers could be indicated by dots so arranged as to make a square as \( \cdot \mid \cdot \mid \cdot \mid \cdot \) or \( \cdot \cdot \cdot \cdot \) They found that square numbers could be determined by the sum of all the odd numbers in a series beginning with one, as 1, 3, 5, 7, 9 -- etc. Triangular numbers were like 3, 6, 10 -- which could be arranged as \( \cdot \cdot \cdot \cdot \) etc. These were found by adding all the consecutive numbers at any place in the series beginning with 1, 2, 3, 4, etc. Other figurate numbers were determined by similar ingenious methods. Today we speak of our numbers as figures and speak of figuring out a result in calculation all due to the old Greek conception of figurate numbers.

Another peculiar problem which intrigued the Greeks and many scholars of the Middle Ages was the search for perfect and amicable numbers. A perfect number was one which was equal to the sum of its factors, as \( 6 = 1 \times 2 \times 3 \) and \( 6 = 1 + 2 + 3 \), and amicable numbers were two numbers which were equal respectively to the sum of the factors of the other. The Greeks knew only one such pair, 220 and 284. The
Arabs also knew of this pair and wore them for friendship talismans. There are only four perfect numbers up to 10,000 so their search for them was similar to one for the proverbial needle in the haystack. Several perfect and amicable numbers were found by mathematicians of the seventeenth century but such problems, today, are found in our puzzle columns and not in our textbooks.

Number theory, however, has added much to practical knowledge even if those who first worked out the theories could not see their applications. Fibonacci, an Italian mathematician, about 1200 A.D., worked out a series of numbers in which each number is the sum of the two immediately preceding it. Later the ratios in this series were found to be the same as the ratio of the famous Golden Section known in Greek architecture. It is said that the musical scale was discovered by Pythagoras through the study of a series, and we know that many modern inventions have been the result of mathematical equations little understood in their day by the scholars who solved them.

The Chinese developed the magic square as their favorite diversion. One of their sacred books tells the story of the first magic square which is said to have appeared to the Emperor Yei, written on the back of a huge tortoise which came up out of the Yellow river. The picture of the Square given in this sacred book indicates that the numbers were made
with little strings of dark and light circles. This was the Chinese way of writing feminine and masculine numbers. The square, itself, is the ordinary, "fifteen," square still worked by American school children. A picture of this first Chinese square can be found in David Eugene Smith's History of Mathematics.3 Magic squares have been the study and recreation of many great men since the old Chinese emperor's experience, and many ingenious squares and other geometric forms have been made with number combinations. Benjamin Franklin was very fond of pondering over magic squares and succeeded in constructing several complicated ones.

Since none of the ancient number systems were adapted to calculations, various mechanical methods had to be devised. The most primitive of these was finger reckoning or counting fingers which were raised or lowered. Pebble calculating was another method which eventually resulted in a formal calculating device called the abacus, but we will first trace the finger reckoning as it developed in Europe after the Fall of Rome. After the great achievements of the early civilizations, it is hard to imagine the utter depths of ignorance which prevailed among the masses of people in Europe during the centuries between the Fall of the Roman Empire and the spread of Arabic culture after the Crusades. People went back to finger

reckoning as the popular means of communication in trade. The buying and selling took place at great fairs, held in the larger cities where there was some protection for the traders against the prevalent banditry and an opportunity to mingle with merchants from various countries. A regular system of number signs gradually grew up and became standardized. It resembled a deaf and dumb alphabet made with the separate hands. The numbers below one hundred were made with the left hand and larger numbers with the right hand. By combinations of the two, numbers into the thousands could be shown but the larger numbers were not commonly used. For many centuries boys who were destined for trade were trained in this skill just as children are trained today to use a typewriter or play a musical instrument.

In 710 A.D., The Venerable Bede, a learned monk of Jarrow, England, wrote a treatise on finger reckoning. Since he was a monk and not likely connected with commercial trade, it is significant that he knew all these hand symbols. They were evidently common knowledge and considered of great importance throughout Europe. They continued to be considered as the chief method of calculation until the fourteenth century even though counting boards were also in use by that time. Finger reckoning with these ancient symbols is still common in the market places of the Near East countries where peasant people of many tongues mingle in trade. A picture of these
finger numbers may be seen in Miss Sandford's history. One of the chief advantages of this number system in those early days was the fact that no form of writing material was necessary. The monks used the precious parchment, but books were very expensive, and the poor had neither books nor writing materials. Waste paper baskets were not needed at all.

The most ancient calculating machine was the abacus. It was used by many different races and its origin cannot be attributed to any one, but its importance in establishing the decimal system cannot be over estimated. Modern calculating machines fill one with amazement at the computations which they can perform, but the only skill required by the operator is dexterity and the ability to read the numbers involved. The inventor of the machine did the thinking which was required for the calculation and expressed his thoughts in the perfection and arrangement of the mechanical parts. It is hard to realize that this same general process took place back in the dawn age of history even though the resulting device was limited to very simple calculations. These first machines served the same purpose as those we have today in relieving people from the painful process of mental labor, and making the training of the hand the most necessary skill.

Since the word abacus means, "dust board," the first abaci were probably made by drawing lines in the sand or dirt.

The purpose of the lines was to provide a way of indicating the decimal value of the natural number symbols placed in the spaces. Pebbles probably took the place of the marks because they could be more easily changed about and counted. The Romans had little marbles called, "calculi," for counters. This word means pebbles and is the origin of our word, calculate. None of the early forms of Egyptian abaci have been preserved but we know by ancient writings that they were used in Egypt and probably in the Asiatic countries as well. The Romans had stone or metal abaci with long and short vertical grooves engraved in the plate. These grooves contained little balls which represented the units and fives of our decimal system, the short grooves containing one ball for five and the long grooves four balls for the units up to five. The order of the grooves changed from right to left as the positions in our number system do today. Other grooves at the right represented the halves, thirds, and fourths, and the twelfths or, "uncia," which were used in measurement.

Little is known of the Egyptian and Greek abaci but a reference in the writings of Herodotus tell us that they both had vertical lines on which the Greeks counted from left to right and the Egyptians from right to left. The Chinese used rods for counters which probably accounts for the form of abacus they developed. They set the rods in a frame and threaded movable disks upon these spindles. This Chinese
The abacus is still in use with slight modifications in form, in many parts of the Orient. The suanpan of the Chinese is large, and has a horizontal bar across the vertical spindles to separate the unit disks from those representing five. It has five counters below and two above this bar. The disks are pushed toward the bar as they are needed. The Japanese use the soroban, a smaller abacus much like the suanpan. The Russian abacus is like the old Greek form. It has no counter for five; the spindles have no horizontal bar and each spindle has nine disks upon it. Oriental people are extremely dexterous in manipulating these abaci and still perform long and complicated computations upon them, obtaining results more quickly than most people can do with figures.

When the abacus replaced finger reckoning in medieval Europe it appeared in a different form, but with the same general plan as the Roman type. It was a ruled board or table on which the operator used loose counters something like modern poker chips. There were usually five horizontal lines on the counting board which looked very much like an enlarged musical staff. Beginning at the bottom line the lines counted for units, tens, hundreds, thousands, and ten thousands. The spaces were fives, fifties, five hundreds, and five thousands. The counters were placed on lines and spaces and shifted about as it was necessary in the calculation. They were made of various materials, but the finest were made of gold, silver
It was considered quite the style to give fine counters as gifts just as we often give silver spoons or pieces of jewelry. Even the kings had their counters in those days.

England and Germany used the counting tables much later than the Mediterranean countries for the Arabic influence brought the Hindu-Arabic numerals into use in the Latin countries. Bank clerks and merchants in England sat behind their lined tables and conducted their business with the counters. Hence the counter in the modern store is so named. Also the terms, "to borrow," and, "to carry," used in subtraction and addition are direct relics of the actual carrying of the counters. The Latin word, "notae," which was used for the counters gave us the words, note, and notation. Line reckoning was taught in the English textbooks on arithmetic until the eighteenth century and occasionally it appeared even later.

As long as the old Roman numerals were used for bookkeeping the counting board was necessary. The invention of printing and the greater distribution of cheap books finally caused the disappearance of the counters, because people adopted the Hindu-Arabic numerals when they became better informed. Slates then became the popular material for computation and they survived in the schools both in England and America until the end of the nineteenth century. Cheap paper put an end to their use.
Miss Florence E. Yeldham has made a special study of the abaci and counting boards which were used during the Middle Ages. Her book, The Story of Reckoning in the Middle Ages, tells how computations were made upon them and shows many pictures of the various kinds. We cannot go into detail concerning these methods but it is most enlightening to attempt the use of a suanpan or counting board. After such an attempt our nine number symbols and the zero look like old friends whom we do not care to desert. Number superstitions, finger reckoning, the abacus, and reckoning on the lines have all left their traces in the words and processes of mathematics, but they have served their day and been superseded by more efficient and modern methods.

CHAPTER VII

TYPES OF NUMBERS AND THEIR USE IN CALCULATIONS

In the beginning of number symbolism, the concrete number word always represented the result of counting. People counted only what they could see or touch. Hence, the first step in mathematical thinking was the concept of number apart from the object, or the abstract number as contrasted with the concrete. In this early stage of development no other type of number was recognized except the whole, or what we now term the positive integer.

The Chinese, before the Christian era, recognized the two types of numbers, the positive and negative, and indicated them in computation by use of red rods for the former and black rods for the latter. This is an exact reversal of our modern use of red and black ink in bookkeeping. It is an exact reversal of our modern use of red and black ink in bookkeeping.

The Greeks, however, recognized a law of signs when multiplying two algebraic differences such as \( x - y \) multiplied by \( x - y \) as \( x^2 - y^2 \). Diophantus, in 275 A.D., solved equations which had negative roots but discarded these results and called them absurd. The Greeks recognized a law of signs when multiplying two algebraic differences such as \( x - y \) multiplied by \( x - y \) as \( x^2 - y^2 \).

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presumed that the ancient Chinese merchant was, in the black, instead of, in the red, during a financial depression. The Chinese writings, however, fail to name the negative number as a distinct type until a much later period.

Recognition of negative numbers by the Hindus is definitely recorded as early as the seventh century, but they did not employ the minus sign. Their symbol was a dot placed over the negative number. The first use of plus and minus signs seems to have been in warehouse records where they were marked upon containers to indicate a shortage or excess of the contents as compared with the regular standard weight.²

Negative numbers were not recognized by Europeans as having meaning other than such concrete examples until about the beginning of the sixteenth century. Various names were used at first to designate the two types such as true and fictitious, positive and privitive, abundant and defective, and affirmative and negative. Today, they are positive and negative or simply plus and minus numbers. Great mathematicians of the Middle Ages who helped to develop the idea of the negative number included Stifel, Fermat, Harriot, Vieta, Descartes, Hudde, and others, all of whom lived in the period from the fifteenth to the seventeenth centuries. The + and − as symbols for the positive and negative numbers were not established in their

² Ibid., p. 399.
present form until the seventeenth century.

Irrational numbers or numbers which cannot be given an exact numerical value were not recognized as a type until the time of Pythagoras. According to the philosophy of his secret brotherhood, the universe was based upon number proportions, and when these men discovered that the diagonal of a square could not be numerically measured, it upset their whole system of philosophy. Tradition tells us that the cult murdered the first man who dared to say that this simple geometric distance was numerically irrational. Many irrationals were later discovered in the relations of geometric forms and they were gradually acknowledged as a distinct type of numbers. The most interesting of all ratios to the ancient and medieval world was the ratio of the diameter to the circumference of a circle. All races tried to find its exact value, but never succeeded. It gave rise to the famous problem of the Middle Ages, the attempt to find a square with the exact area of a given circle. The efforts to solve this problem were continued until the value of the ratio known as pi was computed to 707 decimal places. Not until the nineteenth century was it definitely proved an impossibility. The Greek letter symbol for this ratio, \( \pi \), came into use about the beginning of the eighteenth century.

Fraction numbers as distinguished from whole numbers have

\[3\] Vera Sandford, op. cit., p. 183.
been recognized since the beginning of history, but great difficulty was experienced by ancient people in expressing them. They have been called part numbers, ratios, artificial numbers and broken numbers, the last name being responsible for the word fraction, which is Latin for, "broken." The Egyptians used the unit fractions or those having the numerator, one. The only known exception to this usage was the fraction \( \frac{2}{3} \) for which they had a special symbol. To aid in computation, they had developed complicated tables of unit fractions which they must have memorized as we do, the multiplication tables. Fractions which involved numerators greater than one were considered as ratios and the tables gave their value in unit fractions. Such an example, as given in the Ahmes Papyrus, is cited by Smith as follows:

\[ 2 : 43 = \frac{1}{42} + \frac{1}{86} + \frac{1}{129} + \frac{1}{301} \]

Multiplication tables wouldn't appear at a disadvantage if compared with a long table of such fractions. Unit fractions were also developed by Europeans during the Middle Ages under the name of simple or partial fractions. Algebras, even in the last century gave exercises in separating fractions into simple or unit fractions.

The Greeks had great difficulty in expressing the fraction because of their cumbersome number system. They used alphabet

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numerals and had to resort to what we might call accent marks to designate numerators and denominators. They also distinguished the latter from the former by repeating the denominator numeral. Using our letters as they did theirs, we might write \( \frac{3}{4} \) as c'd'd". Sometimes they avoided symbolism by writing out words as three-fourths or by writing, "in part" between the numerator and denominator. This word method was also used by the ancient Chinese.

The Romans avoided fractional notation by giving names to the fractional parts just as we do in measurement tables. Instead of \( \frac{1}{3} \) yd, we say a foot, or instead of \( \frac{1}{12} \) ft. we say, 3 1/12 inch. Their, "uncia," meant one-twelfth and on this unit they based their fractions for common use. They cared nothing for mathematics except in its most practical applications and added nothing to its development or symbolism, except the terminology which came from the long use of Latin as the language of written literature. The words, numerator and denominator are illustrations of such Latin terms. The Babylonians were probably the first people to use fractions with various numerators, but we know very little about their method of writing and using them. They made extensive use of tables for both fractional divisions and multiplications. The Hindus used a general form for the fraction and it is probably from the Hindu-Arabic influence that our modern common fraction has its present form. A Hindu writer of the seventh century,
named Brahmagupta, wrote one number over another to indicate a fraction. A few centuries later the Arabs used the same position but inserted a bar between the numbers. This form, however, was not generally used in Europe until about the seventeenth century and at that time there was still much variation because of the difficulty in printing the fraction. Even as late as the last quarter of a century there has been controversy over the method of making the bar -- some preferring an oblique bar and others, the present form with the horizontal bar.

Sexagesimal fractions were those in which sixty and its powers were used for denominators in the same way that ten and its powers are used in decimal fractions. They were first invented by the Babylonians who also used a whole number notation based on sixty. How the idea originated is conjecture, but since their priests practiced astrology, it is probable that the supposed length of the solar year as 360 days had something to do with it. On the other hand, the number, sixty, may have appealed to these early astronomers as a base for fractions simply because of its many possible divisors. It is to the Greek mathematicians and astronomers that we owe the development of this idea into a system and to the Romans that we owe the names minutes and seconds for the sub-divisions. The first divisions were called, "pars minuta prima," or the first small part, which was soon shortened to minute. The
second division meant $\frac{1}{3600}$ of the original unit and was called a second from the Latin, "secunda." From these we derive our sub-divisions of the hour and also of the degree. Third and fourth divisions were once used but proved impractical. These fractions were very important in the development of both astronomy and trigonometry because of angle measurement. All through the Middle Ages, sexagesimal fractions were called astronomical or physical fractions as contrasted with the ordinary or vulgar fractions. Common fractions were still called vulgar fractions in the textbooks of the early nineteenth century. Symbols for the degrees, minutes, and seconds of angle measurement were not fixed until the sixteenth century.

Decimal fractions would seem to be a logical continuance of our decimal or, "ten times," system of writing whole numbers, but strange to say, they were not developed for centuries after the use of the decimal base had become fixed. The general use of sexagesimal fractions by the scholars of the Middle Ages may have been the reason for the late development of the decimal fraction form. Several writers of the fourteenth and fifteenth centuries approached the idea of decimal fractions in tables of square roots in which the whole number was written in one column and the figures representing tenths, hundredths and thousandths in another. No decimal point was used. It is interesting to find that the decimal point first appeared in print the same year that America was discovered. It was
used in the Pellos\(^5\) arithmetic of 1492 in connection with a rule for dividing a number by ten or the powers of ten, but no attempt was made to indicate its use as a fractional denominator. It was used merely as a separation mark between the whole number and the remainder. Later, vertical lines were used for the same purpose. In 1530 a German, named Rudolff, used the decimal fraction, with a bar for the decimal point, in computing compound interest, but even then this new idea did not appear to be understood by the majority of mathematicians.

The man who finally explained and popularized the decimal fraction was Simon Stevin, a Flemish scientist and government official. He wrote a treatise on the subject in 1585 and so thoroughly did he explain the plan that his method is still used in our present decimal system except for slight improvements in symbolism. He wrote little figures enclosed in circles after or over each figure of his fractions to indicate their order, a difficulty which we now avoid by the position of the decimal point with reference to the first figure to the right of it. If he had written 27.567 it would have appeared like this, \(27\underline{567}\) or \(27\underline{567}\). Several forms of symbolism were tried out before the decimal point was established and, even now, most Europeans use the comma instead of the dot in writing a decimal fraction. The English writer who was among the

\(^5\)Ibid., p. 238.
first to advocate the general adoption of decimal fractions was Henry Lyte who wrote a book called, *The Art of Tens* in 1619. This was the year before the landing of the Pilgrims, a very short time ago, as the history of mathematics is reckoned. The per cent fraction, while closely akin to the decimal form, is a much older usage. It was an invention of the Italian merchants in its present form, but the idea and practice originated in Roman times as a method of taxation. The words, per cent, mean by the hundred, and this method of counting by parts of a hundred is a logical development of the decimal system. The early symbols were derived from the words and their initial letters. Per cent fractions were first indicated by per $\frac{0}{c}$, then $\frac{0}{c}$, per $\frac{0}{0}$, and finally $\frac{0}{0}$ or %. Per cents appeared in Italian commercial arithmetics in the fifteenth century. Their first use in England seemed to be limited to the fixing of taxes, but after the sixteenth century, per cent fractions came into general commercial use.

Computation with the Hindu-Arabic numerals, both as whole numbers and fractions, has evolved through a long period of experimentation. The use of the Hindu-Arabic numerals was first called algorism — a name supposed to have originated from a contraction of al Khowarizmi, the name of the Arab mathematician who helped to introduce these numerals among European races. Algorism, like Greek logistic, was taught separately from arithmetic for a long time. It gradually
usurped the place of importance as mathematics became a subject for general study in the schools. Then, by some turn of fate, the old name arithmetic took the new meaning, and the new name disappeared from use. Northern Europe and England adopted algorism much later than did Italy, France and Spain and consequently the development of the four processes came largely from these people of southern Europe.

Addition, subtraction, multiplication and division methods antedate the use of the Hindu-Arabic numerals, but were crude and ineffective with older symbols. The early numbers, themselves, were a continuous addition of one symbol after another, but multiplication and division with them were practically impossible. With the Hindu symbols, addition was a little more difficult because of the place value, but it has always remained the easiest of the processes. The earliest written addition shows that only two numbers were added at a time and the addends were repeated for each column addition. For example, to add 735 and 283, these successive steps would have to appear:

```
735
+283
```

```
78
```

```
818
```

```
1018
```

Placing several numbers in columns for addition was introduced about the fifteenth century. At various times, addition was known as summation, composition, and aggregation, and the result at one time was called the product. Later this word was confined to the result of multiplication and sum was
adopted for an addition result. Addition of fractions has always followed much the same plan as our present one, since the reduction of ratios to unit fractions and the use of sexagesimal fractions had fostered the idea of a common denominator. The Egyptians devoted much space in their textbooks to the difficulties of dealing with fractions in computation.

Subtraction was a much more difficult process than addition for ancient people because of the necessity for borrowing value from a figure in one column to add to a figure in another. As long as the abacus was in use, this problem was solved by the literal borrowing and carrying of counters. Since the era of written computation, the problem of how to borrow and how to pay back value has been the cause of much experimentation. We have recently swung away from the so-called "additive" or Austrian method of subtraction which was introduced in our schools several years ago. This method is far from new, however. It was tried out in France during the sixteenth century. The process of subtraction was known as extraction or detraction. The word, minus, is very old and the early symbol for subtraction was just the initial letter, m. 9 - 2 was written 9 m 2.

Another modern innovation which isn't modern at all is the check by, "casting out nines." The first printed arithmetic taught this method and it was in use long before the
time of printing, for checking abacus computation. During the nineteenth century it disappeared entirely from arithmetics but it is back again in the twentieth century books with wider application than ever. This check is based on the fact that 9 is the last numeral and when divided into ten or its powers, will always leave a remainder of 1. Thus if the number 746 is divided by 9 the remainder is \[7 + 4 + 6\] or 17. Dividing again by 9 you have 8 for a remainder. This remainder can take the place of the original number in any computation. For example, the sum of several numbers will have the same remainder as the sum of their individual remainders found by adding the digits in each. The check is even more practical in multiplication and division for the remainders can be used in the place of the original factors in checking a product. Many mystifying tricks can be performed by the use of this property of numbers called the excess over nines.

Multiplication in its first development was simply addition of one number as many times as indicated by the multiplier. We still find boys and girls doing this today when they become puzzled over the position system in multiplication. The Babylonians made tables of products and squares which they memorized or used for reference. The Greeks also made extensive use of tables since there was nothing about their letter numerals to suggest what a product might be. The Egyptians used a process of doubling and redoubling a number until the
addition of certain products should equal the result. For example, 25 \times 27 would be found thus.

\[
\begin{array}{c}
25 \\
\times 27 \\
\hline
\end{array}
\]

Since the starred numbers on the left add to 25, therefore the sum of their corresponding numbers on the right must be the desired product.

\[
\begin{array}{c}
25 \\
\times 27 \\
\hline
127 \\
25 \\
\hline
675 \\
\end{array}
\]

This method was adopted by Europeans and applied to both multiplication and division during the Middle Ages. Doubling and halving were given in old arithmetics as two very important and distinct processes. Complicated methods of multiplication were developed during the Middle Ages by the European mathematicians. Most famous of these was the "Geloshia," method in which the several partial products were placed in a large square divided into smaller squares and triangles. No carrying was necessary in this method and position value was obtained by the diagonal lines. Descriptions and pictures of these old multiplication and division methods can be found in Vol. II of Smith's History of Mathematics.\(^5\)

Pacioli, an Italian writer of 1494 was among the first to use the modern method of multiplication. He also demonstrated seven other methods in his treatise which proves that form and method was far from settled in the early stages of computation with the Hindu-Arabic numerals.

Division or partition has always been considered the

most difficult of processes. Anyone who could calculate long division problems was considered a mathematical genius by medieval Europeans and he might even be accused of practicing black magic. Division was probably first done by repeated subtractions. Halving and doubling were also early methods of the Egyptians and were practiced later by Europeans as well. One popular method in Europe during the late medieval period was called the galley method because the work when completed was supposed to suggest the outline of an old galley ship. This method was also known as the "scratch," method because each figure was marked out when used as we now do in cancellation. Another method of division was to complete the divisor to a multiple of ten or an hundred -- much the same as we now increase a decimal fraction divisor to a whole number. This method was called golden division in contrast with ordinary methods which were termed iron divisions. The method most nearly like modern long division, first appeared in an Arabic textbook of the fourteenth century. The division sign is a late invention. It was probably suggested by ancient fraction form of the Egyptians, who had used a dot to represent the numerator of their unit fraction. The same form was occasionally revived during the Middle Ages. The division sign is not universally accepted in Europe today. Some countries even use the bar between two dots to indicate subtraction.
Division of fractions, usually considered one of the hardest forms of computation, was performed by the Hindus very much as we do it today. Bhaskara, about 900 A.D., gave a rule for such division. European writers called this rule cross multiplication. It is really our method of inverting the divisor without the inversion — much easier and more readily done than the method now taught in our schools. To divide $\frac{3}{4}$ by $\frac{3}{4}$ the rule was to draw cross lines for multiplication, placing the left-right product in the numerator and the other in the denominator. The work would appear like this:

$$\frac{2}{3} \times \frac{3}{4} = \frac{6}{12} \quad (7)$$

The use of $X$ for a multiplication sign probably originated in these guide lines for cross multiplication. This method was very popular during the Middle Ages in the solution of problems by the Rule of Three and the Rule of False Position. The former was a proportion method, the latter a solution by means of approximation and the computation of error between two limits.

Our present method of finding square roots was developed geometrically by the Greeks. Squares and square roots were known, however, by the Babylonians and Egyptians. Clay tablets as early as 3000 B.C. have been found which show tables of squares. The square root sign, called the radical sign, is probably a modification of the letter, $r$, as made.

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7 This example quoted from Smith, *op. cit.*, Vol. II, p. 226.
8 For discussion of, see Sandford, *op. cit.*, pp. 160-163.
during the Middle Ages with the horizontal bar added to take the place of parentheses. It has been used in the present form about three centuries. The words, root and power, were first used by the Arabian mathematician, al Khowarizmi, about 825 A.D. He called the first power of a number, a root because from it, all other powers sprung. Powers were numbers with great strength. He called the second power a square, necessity for counting and measuring. Under very primitive and Fibonacci, a few centuries later, used the cube for the third power. The initials of these words were used as exponents by the Europeans until the sixteenth century. Several kinds of symbolism were then tried out, one of which was to draw a little square or cube as an exponent. This proved unpopular because it was too cumbersome, and finally the little numbers written at the upper right side of a root were adopted generally as exponents.

Such has been the slow evolution of the processes of calculation and the kinds of numbers we use today. Practically all of our symbolism used in computation has been developed since the sixteenth century. Before that time, words and initial letters had to serve the purpose of plus and minus signs, equal was written as a word, and fractions were indicated by accent marks and words. Mathematics has been made much easier since the advent of effective symbolism.
CHAPTER VIII

PRACTICAL APPLICATIONS OF MATHEMATICAL CALCULATION

Mathematics in the beginning was the result of man's necessity for counting and measuring. Under very primitive living conditions these needs were limited to the individual and his immediate group. He had to answer such questions as: How far? How many? How much? How long? Observation of the natural forms about him led to a sense of form and proportion. It is natural to suppose that trade was the first social activity of man. Probably the cave men bartered flints and skins, but no records of such transactions were kept. The first recorded use of mathematical calculation, strange to say, does not seem to have been a commercial use, but a religious one. The calculation concerned the position and movements of the heavenly bodies and the recording of time. Such knowledge was vital to the primitive sun-worshiping peoples in their religious ceremonies. Therefore, the priests became the first mathematicians. Observations of the stars were best made in a clear hot climate; hence we find astronomical calculation first developing in the tropical countries, such as Babylonia, Egypt, India, parts of China and Central America. Angle measurement was one of the first recorded processes and the

Tabe of degrees, minutes, and seconds is the oldest table of denominate numbers still in use. Without the aid of the telescope, these ancient peoples predicted eclipses, calculated the length of the lunar and solar years, knew the relative positions of the constellations at the different seasons, knew the five planets nearest to the earth and the length of time it took for them to complete a circle. Observation of the stars was best made in a clear hot climate; hence we find astronomical calculation first developing in the tropical countries, such as Babylonia, Egypt, India, parts of China and Central America. Angle measurement was one of the first recorded processes and the
table of degrees, minutes, and seconds is the oldest table of denominate numbers still in use. Without the aid of the telescope, these ancient peoples predicted eclipses, calculated the length of the lunar and solar year, knew the relative positions of the common constellations at the different seasons, knew the five planets nearest to the earth and the length of time it took for them to complete a cycle, and were able, by their observations, to fix the time of the solstices and the vernal and autumnal equinox. Had the sacred books of the Mayas been preserved, we might be even more astounded at the extent and accuracy of early astronomical calculations.

The first recorded use of mathematics was distinctly religious among most of the pre-Greek civilizations. During the Middle Ages, by a strange turn of fate, religion was to prove the most retarding influence on its development and use. Practically the only use of mathematics which the Church encouraged during the period known as the Dark Ages in Europe was the rather complicated calculation necessary to fix the date of Easter. Other uses of mathematics were not thought proper for a Christian, because they were devices of the heathen magicians. This was the reason given by the Spanish fanatics for the burning of the sacred books of the Mayas, which probably contained the most valuable astronomical records in existence at that time.
Commercial uses of mathematics developed first in the Tigris-Euphrates valley. The ancient Sumarian race and the Babylonians who succeeded them had developed a complicated system of commercial arithmetic, before the dawn of history. Recent finds in that region include clay tablets dated before 3000 B.C. that indicate the use of bills, receipts, promissory notes, bookkeeping, bank drafts, bills of sale, wage regulations and tax levies. Such a development did not come about in a few years. Applications of mathematics to business practices must have been in use centuries before the date of the existing records. There were great financial families in those days comparable to the Morgans, Rockefellers, and Fords of our day. In the laws of Hammurabi, the great and ruler and lawgiver who lived about 2100 B.C., there are regulations concerning mortgages, loans, interest rates, investments of capital, partnerships, rents, taxation, and inheritance, all of which sound very modern. So far as we know, these people did not use coined money but they used silver by weight. They had two types of numerals for bookkeeping, the cuneiform and the curvilinear, from primitive cavemen. The prosperous commercial nations such as ancient Babylonia have always been prizes sought by war and conquest, and various peoples succeeded in conquering this fertile valley. The commercial activity seems to have waned, and the Phoenicians and Greeks rose to prominence among commercial nations. Then
Phoenicians have left no records of their mathematical usages or achievements. We do know they were great navigators and ship-builders and both of these occupations require accurate calculations. They were also prosperous traders so they must have used commercial arithmetic in some form. The Greeks had many prosperous colonial cities around the Mediterranean Sea and carried on extensive trade but we know very little of their commercial practices. Commerce was beneath the dignity of the Greek scholars and therefore records of business were not preserved through their writings. However, the Greeks developed the mathematics of architecture and art beyond any achievement of the ancient or even the modern world. Their development of geometry and its applications to astronomy and earth measurement was so far beyond their day that we are only now beginning to appreciate its extent. The coining or stamping of money is also an idea attributed to the early Greeks or a pre-Greek people. The earliest known coins were hand made, and appeared about 700 B.C.

Egyptian mathematics was well developed before the European countries had emerged from primitive savagery. Its use in land measurement led to methods of surveying because the overflow of the Nile obliterated landmarks each year. The natural situation in connection with agriculture led to irrigation projects to preserve the water supply. Their worship of the sun god led to astronomical observations, calculations
of the seasons, geometric construction of temples and projects in quarrying and cutting building stone. The great buildings and monuments undertaken and completed by this people at the dawn of history indicate an accurate use of mathematics in surveying and leveling, a knowledge of the laws of the lever and the inclined plane, the laws of construction for permanence and many other mathematical applications of physical science.

The Ahmes Papyrus,¹ the oldest of all known books on mathematics, also indicates extensive use of trade calculations and common measurements. It has problems concerning the apportioning of food, the size of loaves of bread and their price in terms of the price of wheat, the price of poultry and its preparation for the table, the payment of wages and rents, and many other common practices of that day. Egypt was one of the first nations to undertake the storage of grain and the regulation of the food markets. Elaborate use of mathematics was necessary in carrying out such projects, and also in the levying and collecting of taxes to support the strong centralized government which backed such projects. Mathematics and its practical applications have always been a very accurate gauge of the social conditions and mental development of a people. Nowhere is this fact illustrated more clearly than in the history of Egypt.

The Romans were users but not originators of mathematical knowledge. They borrowed the practical mathematics of the Greeks and Egyptians, but never understood or appreciated the finer and more subtle applications made by the first named people. They developed commerce and navigation to a degree never surpassed until the modern era. Roman coins were in general circulation throughout Europe. Trade was carried on with a practical system of weights and measures and the commercial practices known to earlier civilizations were adopted and well organized. Engineering was developed extensively in the construction of aqueducts. The efficiency of the war machine was the first consideration of the Romans, then, as well as now, and this fact fostered the application of civil engineering to road building and bridge construction. This situation in which war fostered the growth of a mathematical science has its modern counterpart in the advancement made by the science of chemistry due to the impetus given by World War requirements.

Commercial arithmetic and scientific applications of mathematics sank into oblivion in Europe during the Dark Ages after the fall of Rome. The trade that was attempted was hampered in more ways than it is possible to enumerate. There was no centralized government to control banditry; petty rulers discouraged trade by exacting heavy duties and encouraging robbery and bribery; every little principality had
different coins than the neighboring city or province. They also had different weights and measures, different taxes and even different forms of mathematical calculation. The problem of transportation made the smallest commercial venture a gamble, with the odds against its success. The city states of Italy were the first to overcome these hampering influences and develop commerce to any extent. Their rise to commercial prestige began about the twelfth century. From that time on, the commercial use of mathematics grew rapidly.

The early textbooks in mathematics which first appeared in Europe, reveal the social and business conditions of the Medieval Age. Many problems are based on the different kinds of money and their values and others show the dangers of shipping goods by land or sea and upon what conditions profit and loss could be calculated. There are problems of aggregation or mixture, problems concerning short time partnerships, interest and loans, problems in compound interest, rents, marine insurance and annuities. Many of these problems are exceedingly complicated. How they were solved without modern symbolism and methods is hard to imagine.

With the possible exception of taxation, the payment of interest is probably the earliest of commercial applications of arithmetic. Interest rates were extremely high among the Babylonians and other ancient peoples. Hammurabi permitted interest as high as $33\frac{1}{3}$ per cent and at one time the Roman
government permitted a 48 per cent rate. Interest rates also depended on the form of the loan. Commodity loans were paid back in kind. Early Babylonian capital was almost altogether in the form of grain, oil, dates or livestock, and, the interest was so many head of cattle or so many measures of the grain or fruit. The word capital comes from the Latin word meaning head, and probably dates back to the time when capital was actually counted by the, "head." Legal rates were fixed at various times, but evaded, then, as they are now. Roman rates under different rulers were \(8 \frac{1}{2}\) per cent, 12 per cent, 25 per cent and 6 per cent.

Compound interest was well known to the ancient world and also common during the Middle Ages. Complicated tables for its computation were made and used, in spite of the fact that the early Christian church condemned the practice of charging interest. This gave the Jews a decided advantage in the banking business, which they have held, consistently, in spite of their persecutions in various European countries.

Interest rates today are controlled by laws, and the charging of compound interest is illegal. It may be offered, however, by banks or investment companies.

Banks, in the early days, were merely places for the exchange of money from one coinage to another or places of relative safety for the protection of money. The latter service was often rendered by the priests in the temples and
after a time, exchange tables were also set up in the temple courts for the accommodation of travelers or pilgrims. Such a situation is described in the New Testament during the life of Christ, when He overthrew the tables of the money changers and drove them from the temple. His caustic comment about the temple being made a den of thieves also indicates a knowledge of their corrupt practices. Bankers were so thoroughly hated during the Middle Ages that their benches at the gates of the cities or the market places were often knocked over and broken by irate customers. This is said to be the origin of the word, bankrupt, for bank comes from bench and a broken bench meant that the banker was temporarily out of business. As political government became more stable, regulation of banking was gradually undertaken. After the organization of great banking companies, laws became more and more stringent but, during the last few years, it has been necessary to add new restrictions to our banking laws and to require insurance of deposits.

Early Roman bankers allowed two kinds of deposits just as we have today. No interest was paid on the checking account, but it was a preferred claim if the bank failed. Bank drafts and bills of exchange have been found on the Babylonian clay tablets, which show that some method of paying bills at long distance has been in use for a long time. Such usages were common during the Greek and Roman period but in Europe,
during the Dark Ages, all such commercial transactions were practically unknown. Credit money can only be used where government is strong enough to guarantee its value. Transportation of coin or other valuables, however, was extremely hazardous and there came a time when the collection of papal taxes enabled Italian cities to again establish an exchange of credit. England had to make these collections for the Church, but it was very hard to send the money to Italy. Italian cities wanted to trade with England, so they agreed to pay the papal taxes at Rome in exchange for goods from England, while the English merchants could use the tax collections to meet the bills due them from the Italian merchants. From this beginning, about the twelfth century, other systems of credit were established between European countries and later, between Europe and the American colonies.

Calculation of profit and loss was a very difficult procedure in medieval Europe. The chances for loss were not limited to poor business judgment. The slow and uncertain sailing vessels had to be taken into account, the numerous robberies on overland routes, the petty bribery of local rulers, the chance for perishable goods to spoil on long journeys and the lack of any protection of the merchant's rights to the property. For this reason, bookkeeping was really a task for an expert and boys were trained in this work as a profession. Problems involving profit and loss
were common in the textbooks of the Middle Ages, but we must bear in mind that these textbooks were not for school children but for adults who were preparing themselves to enter some form of business.

As early as Babylonian days, profits were divided in proportion to the amounts of capital invested. Even then, the partnership was a common method of financing a business venture. In medieval Europe, the partnership was the only legal method of getting a return on capital since the Church would not permit the taking of interest. These partnerships were usually temporary agreements and a business man might be a member of more than one at a time. Thus, he was able to offset losses in some by profits in the others. Agents called factors, were employed by these temporary firms. The factor actually managed the business. Factories, at that time, meant warehouses which were under the control of the factor. Broker is a modern term for an agent but it does not have the same meaning as the older term, factor. The factors were regularly paid a part of the profit before it was divided among the partners, thus anticipating the modern commission.

In the problems concerning rent collections and charges for pasturage, we see reflected the large agricultural interests of the Middle Ages. Common lands, in the days of the Romans, were rented out for pasturage on a proportion basis. Several people sharing the same pasture paid according to the
number of head of livestock and the time they were on the pasture land. Under the feudal barons rents were paid in grain or live stock. Such rental charges were often used as a means of avoiding the papal law against lending money for interest. If property were given as security for a loan, it was said to be rented by the lender until such time as the rent and the loan were balanced by the use of the property. This practice was something like a long-time lease on property and the computations of the exact time when such property reverted to the original owner were carried to great extremes.

Stock companies had their origin about the fifteenth century in the organization of European merchants into large trading companies in which each was pledged to favor his partners in commercial deals and give them credit when asked. Stock originally meant the amount of money which each individual had invested as indicated by a stock or tally stick. The fact that these trading companies helped to finance some of the American colonies as a business venture indicates their strength at that time and their far-reaching schemes to develop trade.

The collection of taxes is probably the oldest of arithmetical computations. Taxes are as old as any form of organized government and they have been computed in many ways. The payment of tribute in the form of produce, live stock, human labor, or human slaves was practiced in ancient times. Taxes
have been the cause of wars and revolutions because of their injustice and cruelty. Their collection was farmed out to vicious men who exacted all they could from the people regardless of the amount they were supposed to collect. Most ancient taxes were direct taxes, but customs were exacted as soon as trade grew to any extent. Roman taxation, which has been a pattern for modern tax laws, included most of the forms in use today -- the property tax, the poll tax, the tariffs, the income tax and the license. The trader was the prey of the tax collector during the Middle Ages. Duties had to be paid at the borders of every little province under the control of a medieval baron. Sometimes a duty was levied for the privilege of merely passing through a country. These taxes were often nothing less than bribes to prevent robbery and confiscation by the officials, themselves. After government protection of trade was established tariffs were levied on imported goods for two purposes, one to help finance the government and the other to protect home industries from foreign competition. The Romans, while exacting in their taxes, gave the people more consideration in the use of the money for public improvements than did other peoples of the ancient and medieval periods. Tax problems did not appear in the early textbooks, probably because the subject was a political one and not popular with the general public.

Most people consider insurance as a very modern
application of arithmetic, but history tells us that certain forms of insurance have been known since the days of Rome. The oldest of these is marine insurance. It was common among the shippers engaged in commerce on the Mediterranean, but rates were high and varied according to the prevalence of pirates. We do not know whether this business was a project of the Roman government or a private enterprise. Lloyds' was the first great insurance company of England and is still the greatest company of its kind. It was organized about the middle of the seventeenth century, as a mutual company, for the protection of merchant vessels and their cargoes but now every known type of insurance may be purchased from Lloyds' of London.

Fire insurance was the next form to gain popularity. Such insurance for houses was issued in England as early as 1681. In America, Benjamin Franklin helped to organize the first fire insurance company in Philadelphia in 1752. It was a fire protection company as well as an insurance company and maintained fire fighting apparatus. The houses insured by the company were marked with metal plates, perhaps, in order that the company's firemen might not make a mistake and save the wrong house.

Life insurance in its present form is a development of the last century, but the calculation of annuities was a very

\[2\text{Description given in Sanford, op. cit., p. 138.}\]
important subject during the Middle Ages. Annuity insurance has become popular in America during the last few years as a form of investment, but annuities in earlier centuries were special grants for services rendered or inheritances in this form. In some instances they were purchased by turning over lands to some duke or baron who wished to increase his domain. Calculation of the future value of these annuities was very difficult before the invention of the system known as logarithms which makes multiplication much easier. There were no tables of vital statistics upon which to base any calculations of the expectancy of life until after the seventeenth century, and no actual life insurance was issued at that time. Mutual insurance through lodges was one of the first forms to be developed, then straight life insurance through organized companies and finally, the many forms now sold as investments. Insurance in all its forms is more popular in America than it has ever been in Europe.

The story of money and the various things which have been used for money would make a book in itself. We can only mention it at this time, and indicate a most interesting and enlightening article from The National Geographic Magazine which gives a detailed story of the evolution of money from ancient to modern forms. Calculation of prices has always

been necessary even in trade by barter, but some standard value by which to indicate price has made it easier to do. Many things have served as money, such as cattle, salt, shells, skins, tobacco and even stones, but metal has proved the most popular. Metal money has not always been coined money. It was first used in bars, rings or links and its value was determined by weight. Copper was the first metal to be used, silver the most widely used. Gold has only been a medium of exchange since the modern era but silver has been used for centuries. The many kinds of money in use during the Middle Ages, created a tremendous problem in computation. One business venture might have values involved which had to be calculated in eight or ten different kinds of money. Modern forms of money are greatly simplified by the use of credit money or such money as currency, checks, bank notes, drafts and letters of credit. In fact, we now have no metal money greater in value than the silver dollar.

The history of measurement is so ancient that we know little of its first chapters. It may be surmised that the measuring of time by the day was the first actual use of a measurement unit. Primitive people still count time by "suns," or, "sleeps." The lunar month was also an unit of time which was in use among early peoples for it was a natural observation of people who lived in the open. It has long since ceased to be a division of time in our calendar.
and most of us today never know when a lunar month begins or ends. The year has remained in the calendar since antiquity and men have calculated its length more and more accurately throughout the ages. The week is an artificial division of time and our present month is also man-made and not a division of nature. Many interesting incidents in connection with man's efforts to measure time and make a calendar may be read in the little booklet called, The Story of our Calendar. We can only mention a few of the numerous units by which men have attempted to measure length and capacity. The idea of surface measurement did not trouble them until the ownership of land became a point of dispute. After all, surface does not exist apart from the solid in the natural world so why should primitive man have found it a problem? Distance, however, was a tangible and necessary measurement and man found units for its expression in the parts of his body. Such units were always at hand and as uniform as anything he could have devised. So the old cubit was the length of a man's forearm, the span, the width of his outstretched hand, the palm, the width of his four fingers and the digit, the width of his thumb nail. These units were in use among the ancient Egyptians and persisted well into the Middle Ages. Naturally should be the length of three: surface units from the ancients.

other units developed as man's needs grew. The foot became more popular than the arm as a measuring stick and the yard gained popularity. The yard was said to be the measure of King Henry's outstretched arm from the tip of his nose to the tips of his fingers or, according to some authorities, it was originally determined by measuring the distance around a man's waist. The pace and double pace were the lengths used for longer distances and the Roman, "mille passuum," or one thousand double paces, was the origin of our mile. Land measure needed a longer unit, and the rood or rod gradually developed to meet this need. The story of the attempt to standardize the rod is, that on the king's decree, the officials of the court were to stand at the church door on Sunday morning and take the first sixteen men, large or small, who came out, have them stand with their left feet in a line, and measure this distance. This was the official rod. Of course, this was an attempt to get an average or standard from the old body measurements. The acre is said to have been established by declaration that the amount of land which could be plowed with an ox team between sunrise and sunset should be the acre or unit of land measure. In the fifteenth century, the English Parliament sought to standardize the inch by ruling that it should be the length of three barleycorns taken from the middle of a dry ear, and laid end to end. Such illustrations could be duplicated in other units of measurement and weight
which have and are still being used but it is not necessary to cite more instances to prove the struggle mankind has gone through in seeking standardized units of measure. The little booklet, already mentioned called, "The Story of Weights and Measures," tells of the plight of medieval Europe when different cities and often different sections of the same city had their own individual systems of weights and measures. Our government now maintains a Bureau of Weights and Measures which enforces the use of certain standard units in every part of the nation. We no longer depend on the king's arm but use the official yardstick carefully copied from the standard yard preserved in Washington.

Even though standardized, our measurement system is still the haphazard relic of primitive methods in the choice of units. The only logical system of weights and measures ever planned and used is the modern metric system based on the decimal scale. All other systems, like the famous Topsy, "just grew." The metric system was devised by the French about the time of the Revolution and was put into practice in France, by decree of the government. Later, all of Continental Europe adopted it, and now, England and the United States are the only civilized countries which still cling to the antiquated measures

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of our ancestors. The metric system is legal in the United States but the government has never enforced its use.

The metric system is based on one linear unit called the meter which is an earth measurement. It required eight years of hard and dangerous toil by the French surveyors to determine the length of the meter which was to be \( \frac{1}{10000000} \) of the distance from the equator to the pole. The divisions and multiples of the unit meter are based on ten and indicated by Latin numerical prefixes for the former and Greek prefixes for the latter. A decimeter is \( \frac{1}{10} \) of a meter while a decameter is ten times a meter. A centimeter is \( \frac{1}{100} \) of a meter, a millimeter, \( \frac{1}{1000} \) of a meter. A kilometer which was used as an official measure in the Olympic games is a thousand meters. The meter measures about thirty-nine inches in our units. All square and cubic units, measures of weight and capacity in the metric system are based on the divisions of the meter and are interlocked so that it is possible at any time to turn one type of measure into another. They are all as easily written or used as our money system because they are all decimal units. Sometime, the boys and girls of this country may rebel against learning tables of measure with no logical connection between the various tables or their divisions and force the actual adoption of the easier metric system.

Applications of mathematics may be found without end,
both in the ancient world and the modern. One of the most recent developments which touches the life of everyone is the use of the graph as a method of presenting statistics. The mere reading of large numbers is enough to discourage further consideration of an important news item, but if the significance of these numbers be presented by proportionate lines, bars, pictured objects, or divisions of a circle, the reader comprehends the meaning at a glance. Picture numbers were the most primitive form of mathematical expression and now pictured numbers have become the most modern way of imparting mathematical information to the general public. As the needs of human society have become more complex, the mathematicians have, in a similar way, anticipated and met these needs with practical applications of mathematics. The uses of mathematics are constantly increasing, and the future will bring with it new problems to be solved. Will the boys and girls of our generation be able to meet and conquer them as effectively as did the great thinkers of the past? No nation or age in the history of mankind has offered greater opportunities for mathematical research and creative thinking than those which confront the boys and girls of our own country today.

The old algebraic problems are of Oriental origin; the Egyptians, Babylonians, Hindus, Chinese and Arabs each copying from the other and adding a few originals. They all loved to dwell upon the mystical properties of numbers and in their
CHAPTER IX

THE EVOLUTION OF ALGEBRA

The symbolism and methods of problem solution which we call algebra, today, are comparatively modern, but the idea of the equation which is the fundamental concept of algebra is as old as the monuments of Egypt. The Ahmes Papyrus, mentioned earlier in this discussion, contains equations with a hieroglyphic symbol for the unknown number. Translating the most famous example, it reads: "Heap, its whole, its seventh, it makes nineteen." In modern symbols it would read $n + \frac{n}{7} = 19$. These early problems must have been solved by a systematic guessing or trial and error method for no rules for actual manipulation of the equation are given in early books. Such problems seem to have developed first from the puzzle idea rather than the practical solution. They must have intrigued the minds of ancient people much as the puzzle columns and crossword puzzles interest people of our generation.

The old algebraic problems are of Oriental origin, the Egyptians, Babylonians, Hindus, Chinese and Arabs each copying from the other and adding a few originals. They all loved to dwell upon the mystical properties of numbers and in their
efforts to find peculiar sets of numbers, they developed extensively, the indeterminate equation or the type which may have more than one solution. Such problems were solved by studying all the properties which the solutions would have to possess and then working out a plan for finding them which we today would call a formula. People still try such problems for the keen pleasure of testing their mental ability. A modern version of such a problem appeared recently in Ripley's, "Believe it or Not" page. The question was, What number will divide by 10 with a remainder of 9, divide by 9 with a remainder of 8 and so on down to the divisor 2. Ripley's answer gave the enormous number 14,622,047,999, but a little thought on the possibilities for such a number shows that there are several solutions, less than the given answer, the smallest of which is the number 2519.

The old puzzle problems were widely copied by the Europeans of the Middle Ages and they appeared as a part of textbooks on algebra until the nineteenth century. Other old problems besides the, "find a number," type which were borrowed from Oriental sources were the partnership problems, the mixture problems, the division of work and cistern filling problems and problems involving combinations. As an illustration of these old types, David Eugene Smith quotes this problem from a Chinese book of the sixth century.
If a cock is worth 5 sapeks; a hen 3 sapeks and one chicken, how many cocks, hens, and chickens, 100 in all, will together be worth 100 sapeks.1

There are Arabic versions of this problem in the work of Mahavira about 850 and much later it appeared in European books in this form:

Twenty persons, men, women, and girls have drunk 20 pence worth of wine: each man pays 3 pence, each woman, 2 pence and each girl 1 penny. Required, the number of each.2

Fibonacci, an Italian mathematician of the thirteenth century gave many of the old verbal problems which the Arabs had used, in settings suited to his day. One of these which is quoted by Miss Sandford has been a perennial favorite with the mathematical puzzle writers. It has appeared in many different forms.

A man went into an orchard which had seven gates, and there took a certain number of apples. When he left the orchard he gave the first guard half the apples that he had and one apple more. To the second, he gave half his remaining apples and one apple more. He did the same in the case of each of the remaining five guards and left the orchard with one apple. How many apples did he gather in the orchard?3

His problems included all of the old types that have been mentioned and several of the type in which complications of the birthright and inheritance laws were stressed. He was

2Ibid., p. 585.
3Vera Sandford, op. cit., p. 221.
also very fond of the series or progression problem. He gives one, already old in his day that we recognize immediately in its riddle form.

Seven old women are traveling to Rome and each has seven mules. On each mule there are seven sacks; in each sack there are seven loaves of bread; in each loaf there are seven knives and each knife has seven sheaths. The question is to find the total of all of them.\textsuperscript{4}

We don't know who changed the wording to the seven wives who were going to St. Ives, but it has persisted to this day. The strange thing about this type of problem was the fact that these old writers utterly ignored the absurdity of adding such unlike things as old women, sacks, loaves of bread and knives.

Verbal problems not only reveal the way the minds of men have sought activity in analytical thinking but they often reveal the general social and economic conditions of certain periods because each writer who used an old problem changed the wording to suit the particular customs of his time. Many of the problems which had some practical application in their day seem ridiculous to us now under changed conditions. The old pursuit problem is an illustration. It originally appeared as the hare and the hound in pursuit; then as the king's messenger overtaking the earlier messenger; the horsemen riding after another; the ships sailing at different speeds or with contrary winds, and finally, the trains starting at different

times and the one overtaking the other. Many of the problems of the Middle Ages reveal the common custom of wine making and wine drinking among the people of Italy and France, while the same problems appear in German books with beer substituted for wine. Laws of inheritance, church, taxes, ways of transportation, trial by lot, pastoral and agricultural life, and the common measures of length and capacity are only a few of the many prevailing customs which are revealed by the puzzle problems of various ages and races.

Such problems also show that ancient peoples could solve both linear and quadratic equations at a very early period in history. This does not mean that they solved them as we would solve them today. Most of their solutions were ingenious forms of trial and guesswork, but in some cases they actually used correct formulas without recognizing them as such. The Hindus used a method for solving the quadratic which was practically the same as completing the square. Several early writers came very close to the statement of the quadratic formula, but the lack of a suitable symbolism prevented its recognition.

The Greek mathematicians of the early period seemed to pay little attention to algebra except to solve certain problems by geometric methods, such as line proportions and the use of squares and rectangles. The first Greek to add to the content of algebra was Diophantes of the Alexandrian school. He was the first writer to consider algebra as a branch of
mathematics and attempt to compile the algebraic knowledge of the past together with important additions of his own in a book. This book appeared in 275 A.D. and was so far in advance of the times that little or no improvement was made in the next thousand years. For this reason Diophantes has been called the father of algebra. We know practically nothing of the life of Diophantes except what is revealed in his work. Several commentaries were written concerning his work but nothing of merit was added to it. One of the commentaries was written by the only woman among the ancient peoples to become famous as a mathematician. This was Hypatia of Alexandria, who lived about 410 A.D. Her life and tragic fate, Kingsley has used as the plot of a novel.

During the long period in which algebra seemed to be making no progress, other developments in mathematics were making it possible for progress to be resumed. The Hindu-Arabic numerals and position system of notation were slowly emerging from their local obscurity in India and the Arabs were carrying them into Spain and Italy along with Syrian translations of the Greek classics. To one of the greatest mathematicians among the Arabs we owe the name of algebra. About 825 A.D., al Khowarizmi wrote a book entitled Al-jabr w'al muqabalah. We shall not ask that this be translated by anyone for in fact, no one knows for sure what it means. Some say it refers to the transposing of negative
terms in an equation, so that they may be combined. Others think that the same word is expressed in two different languages and the word means balance or equality. Whichever idea is correct, we can be sure the name is a direct reference to the equation and the first part of the name "al-jabr," has given us the name, algebra.

Al-Khowarizmi was the second great writer who helped to establish the technique of algebra. He did not use even the few symbols invented by Diophantes but wrote his equations entirely in words. Most of his problems asked for a number which would meet certain conditions because his chief interest lay in the solution of quadratics. One of his problems quoted by Smith read thus: "A square, multiply its root by four of its roots and the product will be three times the square with a surplus of fifty dirkems." His manuscript was translated into Latin in 1140 by Robert of Chester and it became the leading authority on algebra for several centuries.

The third great name connected with algebra is that of Fibonacci, sometimes called Leonardo of Pisa. He lived at the beginning of the thirteenth century and had studied under Moorish teachers. He traveled extensively in the Mediterranean countries and lived for some time in Northern Africa where his father was the head of a large commercial warehouse. There,

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Fibonacci met merchants from all over the world and learned their methods of computation. His skill with numbers was remarkable in his day and he was probably the greatest mathematician of Europe during the early medieval period. He recognized the value of the Hindu-Arabic numerals, and was the first to use them in a textbook of importance. His work called the Liber Abaci appeared about 1202 and later his Liber Quadratorum and Flos. He was the first writer to separate the discussion of algebra and arithmetic and he called algebra the flower of mathematics.

Fibonacci's fame grew to such extent that he was summoned to the court of Emperor Frederick II to engage in a mathematical duel with the king's favorite, John of Palermo. It is said that three very hard problems were given to him to solve in a limited time and that he succeeded in solving all of the three in the specified time. How he did it without any of the modern algebraic symbolism, nobody knows. Anyone who is interested in the problems which John gave to Fibonacci will find them quoted by Miss Sandford.\footnote{Such contests were common during the Middle Ages.}

Vieta, a French mathematician who lived the last half of the sixteenth century was a lawyer by profession, but he made a hobby of mathematical study. His works were published privately for distribution among his friends. He was also a
code expert and made a special study of symbolism. Algebra had developed in a rather haphazard manner during the Middle Ages and Vieta attempted to collect, arrange and standardize the knowledge of algebra as Euclid had done for geometry. His work prepared the way for the calculus of Liebniz which was to radically change the world.

It is hard to realize that in spite of the long history of algebra all its present symbolism has been developed in the past three centuries. A few scattered attempts had been made to use letters for the unknown quantity but no custom had been established before the sixteenth century. Initial letters used for words is probably the origin of our letter symbolism. Various experiments were tried during the sixteenth century in expressing powers of numbers. The use of letters for the root and the Hindu-Arabic figure at the upper right side for the exponent was introduced by Descartes, the great French mathematician about 1637. Symbols for the four processes in arithmetic all seem to have been adopted after the same signs had been first used in algebra. We know that the \(+\) and \(-\) signs were used to denote positive and negative numbers a long while before their use as symbols for addition and subtraction. The multiplication sign has never been popular in algebra and the fraction form is preferred in division. The parentheses as signs of grouping or for separation of factors, were in use by the seventeenth century and the
horizontal bar had become a part of the radical sign. Other symbols such as > < for, greater than, and less than, came into general use late in the eighteenth century although they had been used by Harriot in his writings a century before. The equality sign we now use was first proposed by Robert Recorde, the English physician, in his Whetstone of Witte, an English algebra published in 1557. His reason for such a symbol was that no two things could be any more equal than two parallel lines the same length. It was nearly a hundred years after Recorde's book appeared before the sign finally won enough popularity to establish it as a standard symbol.

Algebraic thinking is a very old process. It is the process of analyzing a problem by logical steps which finally lead to the desired result. The peoples of Egypt and the Orient led the way in this type of mathematical thinking. Algebra was not a practical subject for the common man. It was rather a recreation for scholars. Ahmes of Egypt, Diophantus of Alexandria, al-Khowarizmi of Bagdad and Fibonacci of Pisa are the names which trace its history down through the ages to the sixteenth century. The mathematicians of the sixteenth and seventeenth centuries completed the great work of organizing the subject of algebra and establishing its symbolism. The puzzle problems of old gave way to the puzzles of engineering and scientific study. Algebra is the great tool which permits men to delve into the secrets of physical
What will it bring to light in the future? Probably the secrets of biological science, the laws of weather cycles, the causes of business cycles, and the many other unsolved problems of the present day. Information about all these conditions is slowly accumulating and some mathematical genius of the future will reduce the information to an equation which will explain and forecast the destinies of generations to come.
CHAPTER X

INTUITIVE GEOMETRY AND INDIRECT MEASUREMENT

The earliest use of geometry seems to have been the expression of a primitive desire for ornamentation. The geometric form was also always used with geometric meaning, earth measure. Intuitive means that which is learned from observation. Hence, intuitive geometry includes all that man has learned about the form and measurement of the world about him without the aid of demonstrated proofs. Man must have begun to recognize geometric form long before history began. The sun and the moon were the great bodies in the heavens that gave light and life. They taught him the circle as a symbol of completeness or deity. The polar star fixed his cardinal directions. The symmetry of leaf and limb on the trees, the shapeliness of the mountains, the curves of winding streams, the level of the horizon, and the dome of the sky all added to his ideas of geometric form. The prevalence of parallel and perpendicular lines in the natural world must have intrigued his fancy. How do we know he made such observations? Because he was reproducing such ideas in the decoration of his woven baskets and clay pottery long before writing was known or used. The shapes of the baskets and pottery also
assumed definite geometric form as the skill of the workers increased, and it is by these forms that the archaeologists are able to read the history of many races that left no other clews to their identity.

The earliest use of geometry seems to have been the expression of this primitive desire for ornamentation, but geometric form was also closely linked with nature worship. Such designs as the sun disk, the swastika, the serpent column and many symbols found on the altars of ancient races were magic forms dedicated to the gods. Such geometric designs have been found all over the world and indicate a general spread of culture in prehistoric times or a remarkable similarity of religious ideas. The great stone altars and places of worship seem to have been the first constructions among primitive people to assume geometric form. Such awe inspiring monuments as Stonehenge in England with its enormous stones set in concentric circles, the terraced pyramids of Mexico and the great pyramids of Egypt are only a few of the many manifestations of man's desire to build in a manner pleasing to the gods. Tombs and burial places were also built in definite geometric patterns and as civilization advanced, the palaces and courts of the rich were planned as carefully and geometrically as our most beautiful buildings and parks of today.

Primitive shelters are always found to follow the simple
lines of the well known geometric solids. Among the Indians of North America many such forms were used. The familiar tepee or wigwam of Plains Indians was a cone-shaped shelter. The Pueblo tribes used the rectangular prism form. The Navaho had a rough dome-shaped hogan. The Seminole people used a combination of cylinder and cone, the Eskimo, a hemispheric igloo, and some of the west coast tribes, the simple triangular, "lean-to." The decorative designs used by the Indians in blanket weaving, bead work, basketry and pottery show much use of the triangle. Parallel and perpendicular lines, rectangles, arrow-heads, stepped designs and circles are also commonly used in such designs. The swastika and Greek key patterns are found on the altars and monuments of Mexico and Peru but are not often seen in the designs of Northern tribes.

Among races that were far beyond primitive culture when history dawned, geometry as a mathematical subject, was well developed. The simple rules of measurement were known to the ancient Babylonians and Assyrians and recent discoveries by Professor Neugebauer of Göttingen, Germany, have proved that they knew and used the right triangle theorem, at least in special cases. The Egyptians undoubtedly knew many of the laws and relations of geometry. The book of Ahmes states several rules for the measurement of surfaces, which were probably the results of experiments. No reasons for the rules
appear and some of them are only true for special cases. The area of the trapezoid is given as one-half the sum of the bases multiplied by one of the nonparallel sides. This would only be true for the trapezoid that had one side perpendicular to the bases. He also did not distinguish between the altitude and the side of a triangle, which makes his rule for this figure only true for the right triangle. For the area of a circle his rule is \[ \left(1 - \frac{1}{9}\right) \pi \] which makes the value of \( \pi \), 3.1605. This was a fairly accurate estimate compared to the value, 3, which was used by the Hebrews and other early races.

The right triangle in the 3-4-5 ratio was used in surveying very effectively. By means of ropes knotted and stretched so that this ratio obtained, their surveyors or, "rope-stretchers," were able to establish levels, find the exact east and west line from the north star and perform many feats of indirect measurement. The Egyptians built their temples so that the sun would shine into a hidden recess on the days when it was directly east of the entrance. For this reason, the "rope-stretchers" were very necessary to the architects and builders. A plumb line suspended from the vertex of an isosceles triangle was the instrument they used for establishing a horizontal plane. They also were able to calculate the slope so as to plan extensive irrigation ditches and water gauges. The overflow of the Nile River made the surveying of land a frequent necessity. It is, therefore, not strange and became quite wealthy. This fact probably accounts for his
that Egypt has been called the birthplace of geometry.

Whether such knowledge of intuitive geometry as the Egyptians possessed was common knowledge among other races is hard to say, for so few early records have been preserved. The Hindus and Chinese must have been well versed in measurement formulas and they probably knew the relations of the right triangle, but their knowledge had little influence outside of their own country. It is from the Egyptians that geometry came down to us from that dim past. First, it passed to the Greeks, who were the greatest geometricians of all time, then to the Arabs who preserved the Greek culture. The Arabic literature was translated into Latin and brought the knowledge of the Greeks back to Europe, and finally to America.

It is to the Greeks that we owe the great development of geometry beyond the intuitive type. No one thought of learning new relationships by reasoning rather than observation until the time of Thales who lived about 600 B.C. Thales studied in Egypt and learned all the mathematical lore of the priesthood in that country. He also traveled in Asia and doubtless absorbed much information from the Chaldeans and Syrians concerning astronomy. He is said to have predicted an eclipse in 585 B.C. Very little of his actual work has survived but many stories about him have been told by others. He was a merchant of Miletus, a thriving commercial colony on the west coast of Asia Minor. He was a shrewd business man and became quite wealthy. This fact probably accounts for his
leisure and extensive travel in later life. One interesting story of his business career has come down to us through the writings of Aristotle. He is said to have cornered the market in olive oil, one year, by buying up all the oil presses in Miletus and the nearby islands. When the time came to harvest the crop, he alone had oil presses to rent and naturally, prices soared to the sky. Other stories indicate that he was a renowned scholar and possessed a genial personality with his knowledge. In his later years he founded a school at Miletus, known as the Ionian school, and taught many who became distinguished scholars in mathematics and science.

Chief among the pupils of Thales was Pythagoras, the best known and most outstanding of the early Greek mathematicians. He is supposed to have been born on the island of Samos about 570 B.C. but little is actually known of his life. He left no written memoirs, but later writers give evidence that he was an extensive traveler in the East and studied with great teachers in Egypt, India, Persia and Chaldea. He absorbed the mysticism and philosophy of these Oriental countries as well as their learning. After his travels he returned to Crotona, a wealthy Greek city in southeastern Italy. Here, he gathered about him several hundred wealthy young men who

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1For other stories of Thales see Vera Sandford, op. cit., pp. 5-7.

had time for study and research and founded the first secret society known to Europe. The men were pledged to secrecy and loyalty to him as their leader. They worked out many advance steps in the philosophy of mathematics and especially in geometry. All discoveries were said to have been credited to Pythagoras whether he deserved the credit or not. Otherwise, the luckless member suffered expulsion or even death. The emblem of the order was the mystic pentagon or five pointed star. Pythagoras is best known for having given to the world proof of the law of the right triangle, namely, that the square of the hypotenuse of any right triangle is equal to the sum of the squares of the other two sides. He is also credited with the proof that the sum of the angles of a triangle is equal to 180°, and that equilateral triangles, squares and hexagons will exactly fill the space about a point. He taught that the earth was a sphere and believed number to be the essence of all things in the universe. As did other teachers of that time, he too taught orally, for there was no suitable writing materials for common use. To this fact we probably owe the loss of much of his actual work but we do know that his influence became so great as to alarm the civil rulers. His brotherhood was scattered and he was exiled before his death, but his influence lived on and came to flower in the golden age of Plato and Aristotle.

Many mathematicians of less spectacular fame followed
Pythagoras and continued to add to geometric knowledge. Democritus who lived about 400 B.C. is said to have been the first to demonstrate that the cone is exactly one-third of a cylinder of the same base and height and that the pyramid has the same relation to the prism of equal dimensions. Plato, while not a mathematician, had a great respect for the study of geometry and is said to have had above the door of his Academy these words, "Let no one ignorant of geometry enter here." Aristotle was interested in mathematics and science and is said to have encouraged his pupils to collect material for a history of mathematics. If such a book was ever written, it has not been preserved.

The last great school among the Greeks was founded at Alexandria, Egypt, the city of Alexander the Great. Here, was the greatest university and the greatest library of the ancient world. We could not begin to name all the great scholars and mathematicians who were connected with this school, but their work went far in advance of intuitive geometry. Euclid, the great textbook writer was so well known that his name has been synonymous with geometry for more than two thousand years. He wrote his Elements about 300 B.C. and it was still used as a textbook as late as the nineteenth century. Originally, this great work was written in, "books," or a series of parchment rolls, and it was so

\[Ibid.,\] pp. 103-107.
perfectly organized and complete that practically no improvement was made in his methods until the past century.

Another great scholar of Alexandria was Eratosthenes, the man who first measured the circumference of the earth. He did this remarkable feat by computing the arc of the circumference between Alexandria and the present site of Assouan dam both of which are on the same meridian. This he did by observing the sun and finding the exact difference in time between the two places. Thus he was able to learn the approximate length of a degree on the earth's surface and compute its circumference and diameter. He estimated the diameter within fifty miles of the present measurement. He also computed the distance of the sun and moon from the earth with a remarkable degree of accuracy.

Of all the great Alexandrian scholars, Archimedes is conceded to be the greatest. His work is far beyond elementary geometry but the stories of his discoveries are so interesting that every boy and girl who is interested in science should read them. Apollonius was among the last of the great students of geometry. He added much to advanced geometry but, after his day the light of knowledge gradually grew dimmer and dimmer until Europe, for a thousand years, groped in the darkness of profound ignorance.

4 Ibid., pp. 108-111.
5 Ibid., pp. 111-116.
The Romans did not appreciate the love of the Greeks for geometry. Cicero expressed regret that his people did not care for this intellectual study in these words. "Geometry was in highest esteem with them (the Greeks), therefore none were more honorable than mathematicians. But we have confined this art to bare measuring and calculating." The Romans did, however, observe symmetry in their architecture, and they made use of leveling and surveying instruments, the carpenter's square and the right triangle ratio.

Indirect measurement began with shadow reckoning among the ancient races. The Chinese refer to shadow reckoning in their classics, and Thales is said to have measured the height of Egyptian pyramids by this method. It is easy to measure heights by comparing the shadow of the object to be measured and that of some nearby post. This practice probably suggested the use of reflecting mirrors and the triangle in connection with indirect measurement. Archimedes is said to have set fire to the enemy's fleet in the harbor of Syracuse by means of reflecting mirrors set up on the shore. The Hindus saw in the shadow reckoning the relation of the two sides of the right triangle which we now call the tangent ratio and they made tables of tangents. They also understood the sine and cosine functions which picture wave motion. This discovery of the ratios which can be expressed with the three sides

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6Cicero's Tusculan Disputations.
of a right triangle led to the development of plane trigonometry. In the time of the Greeks the use of trigonometry or indirect measurement led to the solution of spherical triangles and later to the great discoveries in astronomy. The Arabs and the Turks both made contributions to trigonometry during the Middle Ages, but in Europe, people had to say the earth was flat whether they believed it or not. It was only after the invention of logarithms that trigonometry became a practical science in Europe. It has been extremely important in the development of astronomy, civil engineering, and the mechanics of modern machinery.

Both geometry and trigonometry are Greek words, the latter meaning triangle measure. The words parallel, perimeter, diameter, polygon, isosceles, hexagon, pentagon, cone and many others connected with geometry are of Greek origin. The reason for this is easily seen when we understand what an immeasurable contribution the Greeks made to the body of geometric knowledge. Prehistoric people used intuitive geometry for form and design in their pottery and weaving. Geometric form was found everywhere in the natural world and naturally, was copied long before its laws were fully understood. Early measurement was largely guess work but the formulas for the rectangle, the right triangle and the circle were understood and used by the Egyptians, Hindus, Chinese and other ancient races. Geometry soon outgrew the intuitive stage in
Greece and became the greatest of all mathematical sciences in the ancient world. It suggested the related science of trigonometry and together they made possible our modern knowledge of the universe or astronomy, our modern methods of earth measurement or surveying, and our modern designing of machinery or mechanics. The Greeks expressed their mathematical ideas in beautiful architecture, marble statuary and the fine arts. Modern Americans express their mathematics in wheels and cogs of steel which have created great industries and mass production of goods. Yesterday, only the scholars were affected by the study of mathematics. Today, everybody, young or old, rich or poor, is directly affected by mathematics and its innumerable applications. As a fitting conclusion to this story, let us all say with Elizabeth Dice of Texas, "Not all people can major in mathematics, but all can respect it."

7Elizabeth Dice, "What are the Characteristics of the Progressive Mathematics Teacher," The Mathematics Teacher, December, 1934, p. 402.
From the survey of Junior High School courses of study
and textbooks described in Part One the following conclusions
may be drawn:

PART THREE. THE CONCLUSION

CHAPTER XI

CONCLUSIONS AND RECOMMENDATIONS

The preceding chapters of Part Two have merely skimmed
the surface of the possible sources of historical correlation
with mathematics. The conclusion that such a body of material
exists has been shown by these chapters and its correlation
has been pointed out by the chapter headings. The enthusiastic
teacher will find an abundance of material to link any
unit of junior high school mathematics with its historical
past and give it human interest appeal.

The use of historical background has a psychological
basis in the natural curiosity and interest of the junior
high school child. It can be justified psychologically from
the standpoint of teaching technique, as well. Albert H.
Huntington of Cleveland once said,

If we wanted to tie a boat very firmly to the
wharf we would run many ropes out, preferably to snub-
ing posts on either side. So the more connections we
make between our mathematics, in all directions, with
the life of the past, present, and future, the more we
make its permeations enmesh and enthral our pupils.1

From the survey of Junior High School courses of study and textbooks described in Part One the following conclusions may be drawn:

First -- That historical content has been largely neglected both in the curricula and textbooks for the junior high school.

Second -- That the last five years show an increased use of such material in both textbooks and courses of study.

Third -- That no one method of presenting such material has been adopted by authorities, but various methods have been used by different school systems.

Fourth -- That historical background has been successfully used as enrichment in various school systems.

The personal letters from those who have actually given historical enrichment of mathematics a fair trial in their schools indicate that both teachers and pupils are enthusiastic about its use. A paragraph from the introduction to the Lakewood, Ohio, course of study expresses the reaction of this curriculum committee to their experience with historical enrichment. It reads as follows:

We believe that no course in Junior High Mathematics is complete unless it contains definite suggestions all along the way for the teaching of the history of mathematics. It arouses the interest of the pupils more quickly than anything else and gives them an appreciation of the natural growth of mathematics, as the needs for it arise in the development of civilization. The mathematical experiences which the Junior High School pupils are to have must not only satisfy immediate and
reasonably assured future needs, but should also inspire
a few with the desire to continue the study of mathe-
matics either as a vocation or an avocation in order that
the subject may continue to develop as the needs of
society grow. 2

Since the history of mathematics has not been included in
junior high school courses until very recently, teachers have
not heretofore been prepared to teach it unless their own in-
terest led them into the subject. This conclusion is verified
by a remark made by Dr. Moulton of Northwestern University in
a recent article. He said in connection with a discussion of
teacher preparation in the future,

Even in Teachers Colleges where mathematical train-
ing has been comparatively limited, there is a recogni-
tion that more adequate training is needed. It is obvious
that an intelligent evolution of mathematical instruction
cannot be accomplished by persons mathematically unintel-
ligent or uneducated. 3

A final conclusion concerns the meager amount of histor-
ical material now available for young readers. So far very
little has been done in compiling such material for the use of
the junior high school pupil. That this need is being felt is
proved by the appearance of such a series on the market since
this study was begun. In The Mathematics Teacher for March,
1937, this editorial appeared:

The National Council of Teachers of Mathematics is
inaugurating a plan of publishing a series of monographs

2 "Principles Underlying This Course of Study," Mathematics
Course of Study for Grades 7 and 8. Lakewood, Ohio: Lakewood
Public Schools, 19, 932. Section 8, p. 4.
and Mathematics, Feb. 1936, p. 132.
to be known as, 'Contributions of Mathematics to Civilization,' ... The first of the series, Numbers and Numerals -- A Story Book for Young and Old, was written by Professors David Eugene Smith and Jekuthial Ginsburg both scholars in the history of mathematics, and is now ready for mailing. The second monograph on, Great Men of Mathematics, will appear in due time. It is planned to devote the third monograph to, The Story of Measurement. These monographs should be useful not only to teachers of mathematics but also to teachers of the social studies who may wish to use them in their classes as supplementary reading material.  

This announcement, alone, proves that there is a need for such material and a growing tendency to recognize and provide for it. 

In view of the foregoing conclusions based upon this study, the following recommendations seem to be justified.

First -- That historical material should be definitely included in Junior High School courses of study in mathematics.

Second -- That such courses of study should provide suggestions for the guidance of the teacher in presenting historical material; that is, its use should be indicated as suggested activities or introductory background for the several units or it should be used in the form of separate outlined units.

Third -- That more and better compiled supplementary material for younger readers should be made available by writers of mathematical literature and publishers of children's encyclopedias.

Fourth -- That school systems in choosing Junior High School textbooks in mathematics should consider favorably

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those which contain historical references, providing other features are equally satisfactory.

Fifth -- That teachers should be required to have some definite preparation in the history of mathematics before they are granted license to teach Junior High School mathematics.

Sixth -- That teachers of mathematics should give the historical method of enrichment a fair trial in the school room, by enthusiastic and carefully planned presentation.

Seventh -- That all teachers of secondary school mathematics should realize that mathematics is a living, growing body of knowledge which must meet the ever increasing needs of the future and that mathematical creative ability should be encouraged and stimulated in the formative period of youth.

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MISCELLANEOUS


Courses of Study in Mathematics.

<table>
<thead>
<tr>
<th>City</th>
<th>Grades</th>
<th>Years</th>
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<tr>
<td>Chicago, Ill.</td>
<td>7-9</td>
<td>1929 and 1933</td>
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<td>7-8</td>
<td>1929</td>
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<td>7-9</td>
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<td>Detroit, Mich.</td>
<td>7-9</td>
<td>1929</td>
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<td>Fort Worth, Tex.</td>
<td>7-9</td>
<td>1936</td>
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Idaho, State, Grades 7-9, 1932.
Indiana, State, Grades 1-9, 1930.
Indianapolis, Grades 7-9, 1934.
Iowa, State, Grades 7-12, 1932.
Kansas City, Mo., Grades 7-9, 1929.
Lakewood, Ohio, Grades 7-8, 1932-1935.
Los Angeles, Calif., Grades 7-9, 1933.
Louisville, Ky., Grades 7-8, 1933.
Minneapolis, State, Grades 7-9, 1931.
Muncie, Ind., Grades 7-9, 1932.
Newark, N. J., Grades 7-8, 1932.
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Passaic, N. J., Grades 7-8, 1935.
Rochester, N. Y., Grades 7-9, 1932.
Sacramento, Calif., Grades 7-9, 1933.
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