**A Three-Part Study in the Connections Between Music and Mathematics**

A Thesis

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Molly Elizabeth Anderson

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# Acknowledgments

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# Preface

The idea for this thesis originated from my fascination with the studies of both music and mathematics throughout my entire life. As a triple major in Middle/Secondary Math Education, Mathematics, and Music, I have learned more than I thought possible of music and math. In proposing this thesis, I desired to use my knowledge of arithmetic and aesthetics to research how music and mathematics are intertwined. I am confident that the following three chapters have allowed me to develop as an academic in both music and mathematics. This thesis serves as a presentation of the connections of music and math and their application to my academic interests and studies as I conclude my undergraduate journey at Butler University.

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# I

## From Pythagoras to Johann Sebastian Bach:

## An Exploration in the Development of Temperament and Tuning

Whereas music is the language of the soul, math is the language of the universe. Yet, the parallels of these two drastically contrasting subjects are multitudinous. H. Von Helmholtz marvels, "Mathematics and music, the most sharply contrasted fields of intellectual activity which can be found, and yet related, supporting each other, as if to show forth the secret connection which ties together all the activities of our mind."[1] Dissonance and irrationality, fractals and subdivisions, intervallic leaps and modular arithmetic: math is embedded in music. The influence of math upon music is manifested in temperament, which itself resorts to numbers and ratios for an explanation of consonance, dissonance, and pitch relationships. From philosopher Pythagoras to composer Johann Sebastian Bach, temperament is imperative in the study of mathematics in music.

The existence of temperament and a defined tuning system is founded upon the quality of sound and intended meaning of music, and also as a solution to the imperfection of ratios in Pythagoras' foundational tuning concepts. One must first define the technical terms of temperament to understand its development and significance from the thirteenth to the eighteenth centuries, and even today. A *cent* is a universal unit of measure that quantifies the size of an interval in all temperaments. One cent is equal to one 1/100th of a semi-tone (half-step) or one 1/1200th of an octave. There exist other

[1] Fauvel, J., R. Flood & R. Wilson (Eds.), 2003.

tuning units, such as the *meride* of French mathematician Joseph Saveur, which measures to one 1/43rd of an octave. A tuning error, such as the interval from B sharp to C natural in Pythagorean tuning, is referred to as a comma; a *ditonic comma* describes the interval between two enharmonically synonymous notes and measures approximately 24 cents, whereas a *syntonic comma* describes the interval between a just major third and a Pythagorean third and measures approximately 22 cents. The study of temperament is founded upon the study of space between pitches, better known as intervals.[2] Though the sounds of equal and mean tone temperament are most familiar, other temperaments beg attention for the explanation of how modern temperaments came about.

The reason for the foundation and development of temperament originates from Pythagorean tuning, which demonstrates the imperfect construction of octaves by perfect fifths. In fact, if twelve perfect fifths are played in succession on a modern day keyboard, the final note will sound the same as the first pitch, with a mere 7 octaves between. If the instrument's tuning is Pythagorean, the pitch will sound about a quartertone—ditonic comma—from the desired pitch.[3] Because Pythagorean tuning is just that: a tuning. No intervals are tempered, and commas are a result of errors in the ratios of pitches. A Pythagorean major third (four fifths above the original note, transposed down) measures approximately 408 cents, only 8 cents from an equal-tempered third. Table 1 shows an octave of pitches, beginning on C, and indicates the ratios (in unsimplified and exponential forms) and cent measures of each pitch, relative to the base pitch of C. In Pythagorean tuning, the fifth, with ratio 3:2, is the foundation for the ratio of every pitch.

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[2] Barbour, 1951.
[3] Bibby, 2003, pp.13–27.

With this ratio, and the use of the mathematically pure third, all note ratios are formed; all ratios can be computed as powers of two and three.[4]

| Pitch | **C** | **C#** | **D** | **D#** | **E** | **F** |
|---|---|---|---|---|---|---|
| Ratio | 1:1 | 2187:2048 | 9:8 | 32:27 | 81:64 | 4:3 |
| | $3^0:2^0$ | $3^7:2^{11}$ | $3^2:2^3$ | $2^5:3^3$ | $3^4:2^6$ | $2^2:3^1$ |
| Cents | 0 | 113.7 | 203.9 | 294.1 | 407.8 | 498 |

| Pitch | **F#** | **G** | **G#** | **A** | **A#** | **B** | **C** |
|---|---|---|---|---|---|---|---|
| Ratio | 729:512 | 3:2 | 128:81 | 27:16 | 16:9 | 243:128 | 2:1 |
| | $3^6:2^9$ | $3^1:2^1$ | $2^7:3^4$ | $3^3:2^4$ | $2^4:3^2$ | $3^5:2^7$ | $2^1:3^0$ |
| Cents | 611.7 | 702 | 792.2 | 905.9 | 996.1 | 1109.8 | 1200 |

**Table 1: A representation of the ratios and cent measure of an octave based on C**

Interestingly, if this scale built upon C is transposed up and down to create the other twelve chromatic scales, the most distant scales are those built upon $F^{\#}$ and $G^{b}$; though enharmonic in modern-day meantone temperament, these Pythagorean-tuned scales do not contain enharmonic pitches. And so the problem arises: if the exact ratio from pitch to pitch and interval to interval is known, how is it that Pythagorean tuning results in an imperfect transposition? The answer is found in the cent measure of notes on the circle of fifths. The zenith of the relevance of Pythagorean tuning occurred when composers wrote music in the context of modes as opposed to key areas. Because of this composition style and the absence of modulation, commas in Pythagorean tuning did not cause conflicts in consonance, as they would in modulated keys.

For Guillaume de Machaut and other composers in the thirteenth and fourteenth centuries, Pythagorean tuning defined the consonance of melodies and harmonies. Machaut's *La Messe de Nostre Dame* provides modern-day musicians with an understanding of the significance of consonance in Pythagorean tuning. This mass is

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[4] Gann, 1997.

significant in that it is not only the earliest polyphonic setting of the mass ordinary, but also the embodiment of the ideas and theories behind Pythagorean tuning.[5] As shown in Figure 1, the first measure of the *Kyrie* shows two open fifths on D and A, true to the foundational concepts of Pythagorean tuning. The piece explores many modern-day harmonies, but avoids intervals not pleasantly represented through Pythagorean tuning. In a recording by the Hilliard Ensemble, one can hear the differences in consonances and tone quality from modern-day music.[6] It is especially clear in that the cadences avoid the imperfection of the interval of a third: all final sonorities on a tonic chord are of the root and fifth. These drone-like open fifths did not continue to exist in meantone temperament. Not long after the recognition of the exposed open fifths that form the basis Pythagorean tuning, theorists and composers alike desired a temperament in which major thirds sounded sweeter and richer as opposed to out of tune and very much avoided. This persistence resulted in meantone temperament, which has outlasted any other tuning method since its creation in the fifteenth century.

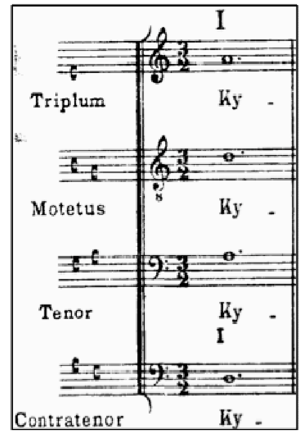


**Figure 1: The first measure of *Kyrie* from Machaut's *La Messe De Nostre Dame***

The temperament method of modern-day pianos and many other instruments finds its roots in the minds of fifteenth-century composers, who desired richer thirds and triads in music. As opposed to the plainchant, monophonic timbre of its predecessor, meantone temperament offered an emotional tone quality with which composers used, and still use, to their full advantage. Meantone temperament values the purity of the third as a

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[5] Paynter, 1992, ch. 51.
Connections of Machaut's mass to Pythagorean tuning are not uncommon, as Pythagorean tuning was the only temperament known at the time.
[6] "Guillaume de Machaut," 3:47.

sacrifice to the perfection of fourths and fifths. With the exception of the Wolf fifth, this slight tempering with intervals is not aurally noticeable. The Wolf fifth, the "perfect fifth" from $G^{\#}$ to $E^{b}$, receives cents from ditonic commas, and its 24-cent imbalance produces a minor sixth sound, which is "too false to be musically useful."[7] This new definition of tuning, as based upon pure thirds and modified fourths and fifths, assigns different cent-measures to pitches, as seen in Table 2.

| Pitch | **C** | **C#** | **D** | **D#** | **E** | **F** |
|---|---|---|---|---|---|---|
| Cents | 0 | 76.0 | 193.2 | 310.3 | 386.3 | 503.4 |

| Pitch | **F#** | **G** | **G#** | **A** | **A#** | **B** | **C** |
|---|---|---|---|---|---|---|---|
| Cents | 579.5 | 696.8 | 772.6 | 889.7 | 1006.8 | 1082.9 | 1200 |

**Table 2: Measures of pitches in meantone as recorded by Pietro Aaron in 1523**[8]

The adjustment of cent-measures also creates alternate thirds and fifths, offering an explanation to common keys in which most music was composed. These "pleasant" keys are said to be the motivation behind the preludes and fugues of Johann Sebastian Bach's *Das Wohltempierte Klavier*.

Why is it then, though many systems of tuning and temperament already existed, that Bach so desired to compose a prelude and fugue in every major and minor key? The philosophy of the day is significant for this explanation. In the Seventeenth and Eighteenth Centuries, philosophers moved away from the ideals of Christianity and toward the model of Greek-inspired Classical Humanism. Whereas Christians worship God with their heart and soul, base their morals upon his ideals (as recorded in the Bible), and wish to spend eternity in the presence and happiness of God, Classical Humanists took on an entirely different view. Much like Grecian perspective, "God was a logical abstraction, a principle of order, the supreme good, the highest truth; God was a concept,

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7 Lewis, 2011.
8 Gann, 1997.

impersonal, unfeeling, and uninvolved with human concerns."[9] It is the voice of reason that allows humans to reach ultimate happiness, not a spiritual relationship with God.

This return to ancient worldviews is echoed in the construction of the forty-eight preludes and fugues of Bach's Well-Tempered Clavier. In comparison, Bach's First Invention, BWV 772, is much like the consistency of Christian ideology, and the Well-Tempered Clavier models the reason behind classical humanism. The two-part form and melodic motives of the Invention are similar to many other pieces of the time, and the steadiness of beauty in the WTC proves the existence of consonance in every key. Bach may have composed the 48 preludes and fugues for the sole purpose of returning to the ideals of Pythagorean tuning methods.

Johann Sebastian Bach's Invention no. 1 in C major (BWV 772) was written in accordance with meantone temperament, the accustomed tuning method of the time in which it was composed. As many inventions do, this piece features two motives and their augmentations and inversions. The "a" motive consists of four ascending sixteenth notes, and the "b" motive contains two descending thirds. These two motivic ideas and their variations combine to create each bar of this invention, as shown in Figure 2.[10] The focus of this piece is primarily upon the intervals of a third, fourth, and fifth, which correlates directly with the tuned intervals in meantone temperament; this system was the sole method of temperament at the height of composition for these short, two-part, contrapuntal inventions. The third is so important that the "b" motive is a pair of

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[9] Loflin, 2004.
This brief history of Christianity and Classical Humanism is important in understanding the motives behind Bach's Well-Tempered Clavier.
[10] Thomas, 2:36.

descending thirds. This focus on near-perfect intervals clearly contrasts the melodic and tonal foci of the preludes and fugues of Bach's Well-Tempered Clavier.

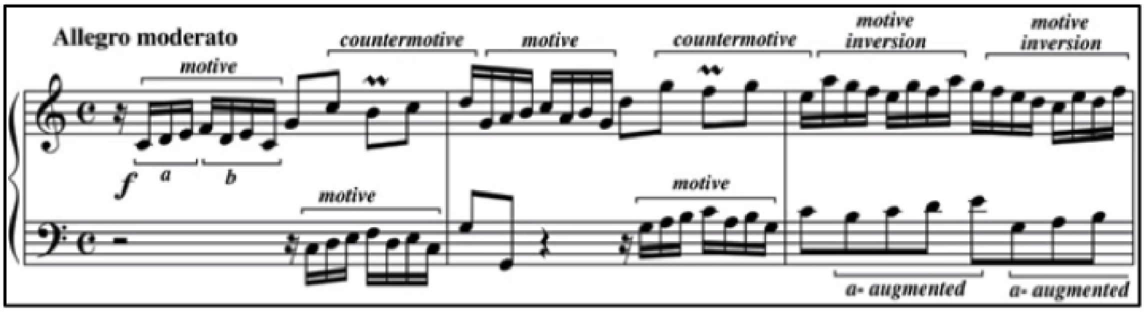


**Figure 2: The first three measures of Bach's Invention No. 1 in C major, BWV 772**

Johann Sebastian Bach composed the preludes and fugues of the Well-Tempered Clavier to prove the existence of liveliness and color in every possible key. The music of Bach and Handel brought recognition to the system of well temperament, though it was founded at the same time as meantone temperament. Through the concealment of the wolf fifth, well temperament allots each key a distinct sound. The greatest of all characteristics of this temperament style is that every key and chord is usable—quite a progressive thought for the time. Pieces composed in the keys of C, F, and G most often sound tranquil and melodic, making them the most accessible keys in which to compose. In opposition, keys with many sharps sound vivid and gleeful, whereas keys with many flats sound solemn and heavy.[11] Composers used these key characteristics to their advantage, composing to the sonic strengths.

The Well-Tempered Clavier recognizes the importance emotion evoked from the tonic key. The subtitle for the piece alone explains the purpose for using well temperament:

[11] Stoess.

*The Well-Tempered Clavier,*
*or*
*preludes and fugues in all tones and semitones,*
*in the major as well as the minor modes,*
*for the benefit and use*
*of musical youth desirous of knowledge*
*as well as those who are already advanced in this study.*
*For their especial diversion, composed and prepared by*
*Johann Sebastian Bach,*
*currently ducal chapelmaster in Anhalt Cöthen*
*and director of chamber music,*
*in the year 1722.*[12]

The first prelude in C major, BWV 846, clearly demonstrates the importance of chords in well temperament. Though C major is arguably the easiest key in which to compose, the establishment of broken chords in this prelude sets a foundation for the following forty-seven compositions in the WTC. German theorist and analyst Siglind Bruhn believes this broken-chord analysis to be true and observes that the movement of chords is simply to ebb and flow in musical tension. Figure 3 shows a block chord analysis of the entirety of Prelude No. 1.[13] As opposed to the focus on intervals in the aforementioned invention, this prelude focuses on the foundation of well temperament in the emphasis on chord development. The piece begins with variations on tonic and dominant chords (as seen in the first two measures of Figure 3) and then moves into a more chromatic midsection,

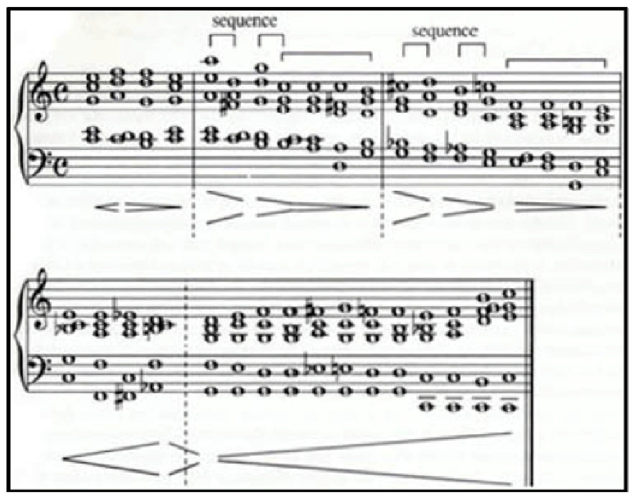


**Figure 3: A block chord portrayal of Prelude No. 1**

[12] Bruhn.
As adapted from Bruhn's analysis of the Well-Tempered Clavier.
[13] Ibid.

before once again returning to the comfort of repeating tonic and dominant chords. It can be argued that this prelude follows that of a sonata—exposition in tonic material and modulates to the dominant, development that brings out various chromatic keys, and a recapitulation returning to the tonic key and the principal "subject" of the piece. Not only is this piece defined by its melodic development and simplicity but also by the tension in the dynamic notation. Though the note patterns are related throughout the piece (with the exception of the last few bars), Bach includes specific dynamic markings with which one may effectively play this piece as meant to be.

Despite the lack of focus upon intervallic relationships and small-scale details, this prelude is incredibly complicated, but on a large scale. In his quest to return to the simplicity of Greek tuning systems, it is possible that Bach complicated the basis of composition. Well temperament, though based upon similar characteristics as Pythagorean tuning, introduces the idea of "emotion" and "response" in music—characteristics which the ancient mathematically based tuning systems did not have.

# II

## The Musical Dice Game and Chance Composition

Multiple composers throughout history created musical dice games, allowing common people, with little or no background in music, the access to the tools to compose a piece of music. Simply, the players of these dice games roll a die (or two), identify the pre-composed measure of music, and copy the measure onto a musical staff. The composition of these games includes the assignment of dice rolls to measure numbers, the measures themselves, and the number of measures completed at the end of the game. The variety of dice rolls does not affect the structure of the type of composition but rather the harmonic structure of piece itself. This compositional style is similar to mad-lib sentence composition; each empty measure has a limited number of possibilities of previously composed measures, much like how the syntax of the sentence limits the word choice in Mad Libs. Johann Philipp Kirnberger, Wolfgang Amadeus Mozart, Franz Joseph Haydn, and many other composers, are accredited with chance composition games.

The origination of the musical dice games is accredited to Johann Philipp Kirnberger's "Der allezeit fertige Menuetten- und Polonosien-komponest"[14] ("The ever-ready composer of minuets and polonaises"), published in 1757.[15] In his dice game composition, Kirnberger writes 96 pre-composed measures for two instruments, each written in treble, alto, tenor, and bass clef, so that any instrument may play the piece. The 16-measure minuet is written in two eight-measure parts; each measure has six

[14] Kirnberger.
[15] Harkleroad, 2006, p. 72.

possibilities (one for each roll of the die), and the first eight measures are repeated. Because there are only six possibilities per measure, this game is played with a standard six-sided die, and each measure has an equal probability of appearing in the final composition. Harmonically, the piece is written in D major, modulates to A major in the middle, tonicizes E minor for two bars, and returns to D major in the final measures. In measures 9 and 10 of the composition, the key of E minor is hinted at with the use of D sharps (leading tone in E minor). This harmonic motion creates a ii chord in measure 10 (an E-minor chord in the key of D major), setting up a ii-V($V^6$)-I progression in the key of D-major. Kirnberger uses compositional elements such as trills, triplets, and passing chromatic notes to add variety to the otherwise harmonically basic measures. What's more is the incredible number of possibilities for this composition—some differing by a single measure and others by multiple measures. In each of the sixteen measures, there are six possibilities for composition, totaling $6^{16}$ = 2,821,109,907,456 possibilities for unique minuets. This, the original dice game, paved the way for other composers to create similar chance compositions, allowing those with no musical background to create their own minuet.

Franz Joseph Haydn's dice game, composed around 1790, offers a seemingly simpler composition method and requires only one six-sided die. This, like Kirnberger's game, ensures the equal probabilities of each measure appearing in the final composition. Leon Harkleroad, author of *The Math Behind the Music*, includes a listening example of Haydn's game played with measures created with only ones rolled, only twos rolled, and then an alternating 1/2/1/2/etc. composition.[16] Instead of creating a minuet, Haydn's dice

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[16] Harkleroad, 2006, pp. 72-73.

game generates a 16-measure trio, modulating from G-major to D-major, briefly features C-major, and modulates back to G-major. Similar to Kirnberger's transition back into D major, measures 9 and 10 of Haydn's dice game frame a tonicization of E minor, which creates the minor ii chord in D major for the recapitulation. Differing from Kirnberger's game, Haydn's composition only offers six possibilities for each measure, but the total number of unique compositions is $6^{16}$ = 2,821,109,907,456—more than Kirnberger's dice game. The tree diagram of Figure 4 demonstrates the possible combinations Haydn's dice game in the first two measures.[17] The player starts at the left-most box and rolls a die. When the first number appears, the player follows the branch out to the proper number, repeating the process until all 16 measures have been attained. This figure only illustrates the possibilities for the first two measures of the piece, and it already has 42 branches!

[17] Harkleroad, 2006.
Adapted from tree diagram on p. 77.

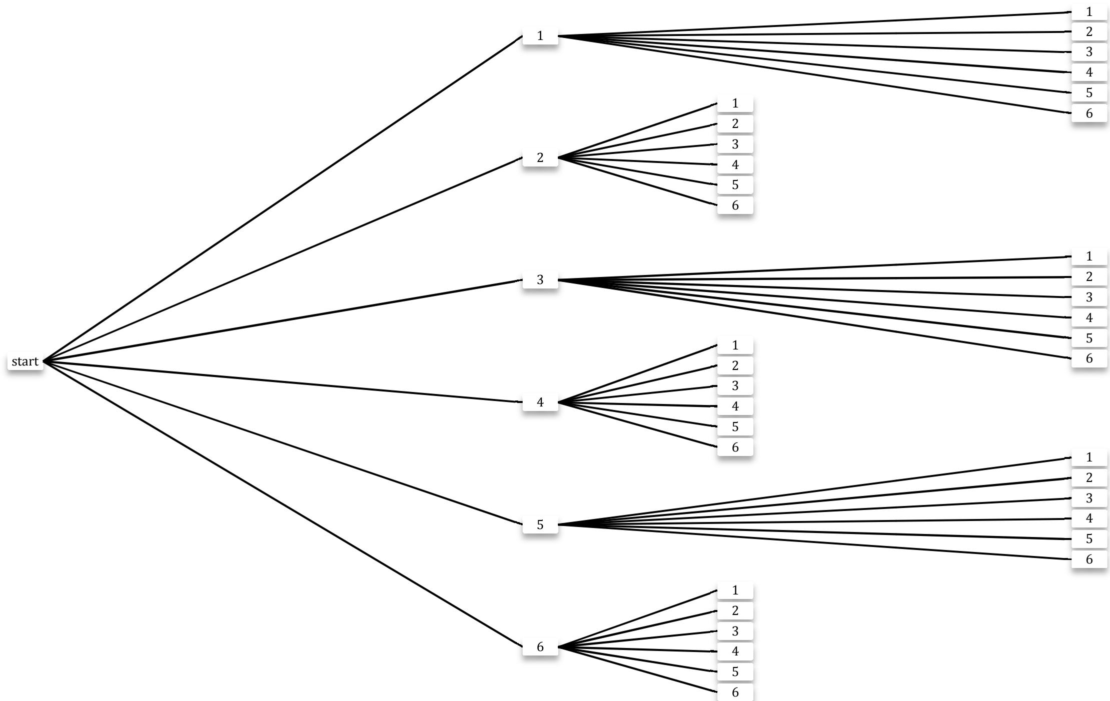


**Figure 4: Tree diagram of the possible combinations for the first two measures of Haydn's Dice Game**

The third dice roll adds six additional branches to each of the right-most branches above, the fourth adds six more onto those, and the process continues for each of the 16 measures. Initially, instead of formatting the game as six lines of 16 measures, Haydn wrote the game as a table, which would assign a measure to the number produced from the roll of the die, and the player would then search through the list of measures for the correct assignment. However, Harkleroad recognized the equal probability of each die roll and reformatted it, making the game easier to read and perform, without re-notating the measures. In this format, the composition is transcribed as 6 rows (one for each roll of the die) with 16 measures each; so if a player were to roll a one for each of the 16 rolls of the die, the first row of Harkleroad's arrangement would be read left to right without jumping to another row.

What is it, then, that distinguishes Mozart's dice game from those of Kirnberger or Haydn? Mozart's *Musikaliche Würfelspiel* requires the player to use two dice and add the two numbers to determine the measure number. The measures coordinate with sums 2 through 12, the lowest and highest of the required two dice. Additionally, the resulting minuet is 16 measures, with the first eight measures repeated. Much like the games of Kirnberger and Haydn, Mozart's *Musikaliche Würfelspiel* is a modulating piece, beginning in C major, modulating to G major in the middle, and returning to C major to end in a perfect authentic cadence. The exquisiteness of the piece comes from the sheer size of it. There are eleven possibilities for each of the sixteen measures, making the total number of unique minuets $11^{16}$ = 45,949,729,863,572,161—over 45 *quadrillion* options.


**Figure 5: An example of the eighth-measure pattern**

The harmonic chord structure of Mozart's dice game is not complex, but the methods of modulation are intriguing. As previously stated, the piece begins in C major, following a I, I, V, I structure in the first four measures, regardless of the number rolled on the dice, to solidify the tonic. In the fifth measure, the piece has already modulated to G major by the way of common chord modulation. C major is not only a I chord in C major, but also a IV chord in G major; thus the piece modulates in measure 4 and features a dominant chord on D in G major in measure 5. The eighth measure set in Table 1 has two left-hand parts written, so as to function harmonically as a modulation back to C major, or as a continuation of G major, as seen in Figure 5. Looking forward to a modulation back to C major, the first two measures of the second table continue in G major, featuring a V and

$I^6$ chord. This is again a common chord modulation, as a $I^6$ in G-major is a $V^6$ in C-major. This recapitulation into C major closes the piece, as a finale to the dice game.

The seventh measures in each table are quite interesting and are the only two in the whole piece to explore more than one chord area within the measure. In Table 1, measure seven features three unique chords in G major, and in Table 2, measure seven (measure 15 overall) features a secondary dominant, perhaps hinting back at the G-major section in the middle of the piece. Taking a closer look at the seventh measure in Table 1, the chords modulate from a root position II, to a second inversion tonic, to a V, which anticipates the I chord in the eighth measure. Though these are the most complex of the measures in the entire dice game, their accelerated harmonic motion is not unusual, as they lead to a cadence in the eighth measure. The seventh measure in Table 2 features both a $ii^6$ and a V chord in C-major, leading to the final cadence in the following measure and following the same compositional pattern as the harmonic motion to the cadence in the eighth measure in the piece.

The first measure that is identical in all eleven options is the last measure in the first table. This measure is also the repeated measure, so it must have a harmonic structure that works for not only a C-major chord progression but also a G-major chord progression. While each of the eleven measures have a root-position G-major chord in the right hand, Mozart created two unique descending eighth note patterns in the left hand so the measure would fit both key areas. The first ending features an octave jump from G2 to G3 and then descends on the major scale to D3, anticipating the cadence to happen on C3. This, of course, happens as it repeats the first measure of the piece, all eleven choices of which are some variety of a root position C-major chord. The second ending jumps from G2 to B3, and descends the G-major scale to E3, predicting the cadence to occur on D3, the dominant of G major. The last measure in the second table has eleven options, but taking a closer look, all but one measure are the same: a quarter note on C5 in the right hand with a descending C3-G2-C2 eighth note pattern in the left. The unique measure—a result of rolling an eleven, which has only a *2 in 36* chance—consists of two eighth notes jumping from C5 to C4 and then an eighth rest in the right hand, and a quarter note on C3 descending to an eighth note on C2 in the left hand.

These two-measure sets toy with the math and the possible *unique* outcomes of minuets from this game. Instead of having eleven options for the last measure of table one, there really is only one option written—identically—eleven times. In the final measure of the piece, there are eleven options, but ten of these measures are identical. The uniformity of these measures alters the number of unique minuets, as can be seen in Figure 6. Instead of the first table having $11^8$ different possibilities, there really are only $11^7$, because the last measure only has one option. In the second table, instead of $11^8$ possibilities, there are $11^7 \cdot 2$, because there are only two distinct outcomes for the final measure of the piece.

**Table 1**

| Possibilities | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | Total |
|---|---|---|---|---|---|---|---|---|---|
| Not counting repeated measures | 11 | 11 | 11 | 11 | 11 | 11 | 11 | 11 | $11^8$ |
| Counting repeated measures | 11 | 11 | 11 | 11 | 11 | 11 | 11 | 1 | $11^7 \cdot 1$ |

**Table 2**

| Possibilities | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | Total |
|---|---|---|---|---|---|---|---|---|---|
| Not counting repeated measures | 11 | 11 | 11 | 11 | 11 | 11 | 11 | 11 | $11^8$ |
| Counting repeated measures | 11 | 11 | 11 | 11 | 11 | 11 | 11 | 2 | $11^7 \cdot 2$ |

**Figure 6: Possibilities of measures from tables 1 and 2 from Mozart's *Musikalische Würfelspiel*, with and without the repeated measures**

Reviewing the original math, and accounting for the repetition in measures 8 and 16, there are not over 45 quadrillion distinctive possibilities for the minuet, but instead

$(11^7 \cdot 1) \cdot (11^7 \cdot 2)$ = 759,499,667,166,482, which is just over 759 trillion possibilities. In the big picture, this is only 1/60 of the original number of possibilities calculated, but still a monumental number.

It is difficult to conceptualize a number this large, but think of the duration of a 16-measure minuet. If this minuet were played—approximately 72 beats per minute—the whole piece would take around 1 minute. This is assuming there is no pause or *ritard* in the piece, 24 measures of 3/8 time would take exactly 60 seconds. If a musician were to play straight through every one of the over 759 trillion derivations of the piece without breaking, it would take him or her just under 1.4 *billion* years. Considering the piece was composed in 1787, just 227 years ago, the probability of composing an identical minuet to a given composition (from this game) since 1787 is very slim—in fact, (were the possible measure choices equally probable) it is 1 in 759,499,667,166,482.[18] It is incredible to think that 176 measures of simple harmony can create a piece that has the potential to last over multiple *eras.*[19] Of course, not all of these 759 trillion possibilities are equally as likely, as some dice sums are more likely than others. In figure 3, the repeated outcomes are seen on the diagonal, and it is clear that rolling a sum of a 6, 7, or 8 is far more likely than a roll of a 2, 3, 11, or 12, which each appear at most twice on the table. Though it does not appear that Mozart took into consideration the probabilities of

[18] This number, 759,499,667,166,482, has been found by multiplying 759,499,667,166,482 by 60, the number of seconds it would take, at a constant tempo of 72bpm. I then divided by 60 (resulting in the number of minutes), then divided again by 60 (resulting in the number of hours), divided by 24 (resulting in the number of days), and then finally divided by 365.25 (resulting in the number of years, and .25 accounting for the leap year that occurs every 4 years).

[19] 100,000,000 years.

dice rolls in the harmonic structure of the piece, there is certainly opportunity for the mathematical chance to overpower the compositional structure.

| | **1** | **2** | **3** | **4** | **5** | **6** |
|---|---|---|---|---|---|---|
| **1** | 2 | 3 | 4 | 5 | 6 | 7 |
| **2** | 3 | 4 | 5 | 6 | 7 | 8 |
| **3** | 4 | 5 | 6 | 7 | 8 | 9 |
| **4** | 5 | 6 | 7 | 8 | 9 | 10 |
| **5** | 6 | 7 | 8 | 9 | 10 | 11 |
| **6** | 7 | 8 | 9 | 10 | 11 | 12 |

| Roll | Probability (out of 36) |
|---|---|
| 2 | 1 |
| 3 | 2 |
| 4 | 3 |
| 5 | 4 |
| 6 | 5 |
| 7 | 6 |
| 8 | 5 |
| 9 | 4 |
| 10 | 3 |
| 11 | 2 |
| 12 | 1 |

**Figure 7: The possible outcomes for a roll of two, six-sided dice and their probabilities.**

Although Mozart did not account for the probability of dice rolls, there is potential for accommodations to the idea of the dice game to account for probabilities. Overall, the harmonic structure of Mozart's dice game is not complex—the melody is not unusual, the chord progression features an abundance of I and V chords, and the modulation and cadences are expected. Referring to the second table in Figure 7, the probabilities of a roll of two dice resulting in a 5, 6, 7, 8, or 9 are each higher than the other possible rolls. In a dice game accounting for the probabilities of dice rolls, heretofore referred to as $DG_P$, these rolls with higher probabilities would take on expected compositional occurrences throughout the measures, much like the measures of Mozart's dice game. For $DG_P$, there are many options for differentiation as a result of rolling a sum with a lower probability. $DG_P$ could alter the harmonic structure of the piece with uncommon chords—such as iii, vii, Neapolitan, and augmented sixth chords—as appropriate in the harmonic progression of the piece. The possibilities for accommodations of probability are not limited to chords. In fact, every musical aspect of $DG_P$ could be altered depending on each roll of the dice. Even and odd rolls could determine the dynamics of a measure; prime numbers could determine a transposition of a measure up or down a set number of octaves; endless possibilities exist for $DG_P$ and its incorporation of the probability of the dice rolls. These changes would not only affect the performance of the piece, but just how many possibilities there are for $DG_P$. The total number of unique compositions of $DG_P$—an exponentially larger number than the result of Mozart's dice game—would have to account not only for the chords and their respective (weighted) probabilities, but also for the probabilities of the other musical elements altered as a result of the new game.

It is clear that these dice games have left their mark on many composers, in addition to paving the way for chance in musical composition. Even today, there is a smart phone app that produces midi files in the style of Mozart's dice game.[20] Composer David Cope is a developer of a collection of computer programs called "Experiments in Musical Intelligence" (EMI), which "produces original works in the style of a particular composer by recombining atomized musical quotations derived from that composer's works."[21] In a way, EMI is the 21st-century version of Mozart's *Musikaliche Würfelspiel*. Instead of reorganizing 176 measures, EMI takes a more complicated route in chance composition. In short, it scans various compositions by a single composer and, at the click of a button, can produce a piece that has similar harmonic and compositional elements as an original work by the chosen composer.

The process of EMI is broken down into six steps in Loy's *Musimathics*, in a chapter entitled "Next Generation *Musikaliche Würfelspiel*." First, the EMI user must choose a

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[20] https://itunes.apple.com/us/app/mozarts-dice-game/id311413994?mt=8.
[21] Loy, 2006, pp. 400–406.

selection of works from one specific composer—such as Bach chorales or Mozart sonatas—these works should demonstrate consistent composition styles so that the EMI may detect musical elements specific to that composer. The EMI program then "performs a lexical analysis based… on Noam Chomsky's theories of the structure of natural languages, and… a harmonic analysis of the works based on the ideas of Heinrich Schenker."[22] In short, the EMI is examining the musical selections on both microscopic and macroscopic levels, assigning numbers to notes, on what beat they fall, how long they last, at what dynamic they are played, and what notes precede and follow them. To take this one step further, the EMI identifies the characteristics of the pieces unique to the composer, and then looks at the fundamental elements of composition from the various works. The fifth and sixth steps take these analyses and observations and generate a "random" composition, following musical expectations and imitating the style of composition of the selected composer, which is either produced as a midi file or sheet music. From the numbers assigned to each and every note, to the numbers assigned to the characteristics of composition, EMI creates a recombination of this data and composes a brand new, old piece.

Using EMI to compose music is relevant to the differences in artificial and human intelligence. Loy questions the intelligence of EMI and adds, quoting Alan Turing, "If we can't distinguish between an intelligent person's choices and a computer's choices, then it is reasonable to say that the machine is behaving intelligently."[23] In a way, EMI takes elements of music, which Mozart worked his entire life to achieve, and creates an imitative work in his style in a matter of seconds. Is this artificial *intelligence*, or just

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[22] Loy, 2006, pp. 400–401. Schenker, 1979.
[23] Loy, 2006, p. 403.

compositional forgery? I think the argument is a result not of the advanced capabilities of EMI, but rather the fact that it is *possible* for an EMI Bach cantata to move a listener as much as a true Bach cantata could. It is maddening that a computer-generated piece has the possibility to possess as much meaning, feeling, influence, and "raw" emotion as a work composed over 250 years ago. An incredible example comes from David Cope's interview with Radiolab, entitled "Musical DNA."[24] At about 5:55, we hear an EMI Bach composition, which sounds rigid as a grand piano midi file; we hear the same piece, arranged as a chorale, at 6:30, and it is as moving—in dynamic, harmony, and melodic movement—as an original Bach Chorale. Cope initially threw this piece away, as he was turned off by the rigorousness of the midi sounds, and now admits it to be one of his favorite compositions ever created by EMI. Additionally, Cope admits that EMI is "messing with some pretty powerful relationships… and doing so in a mechanical way."[25] Cope realizes that EMI frustrates avid classical music listeners—making them question, for example, if Chopin is just "clichés strung together," as EMI Chopin sounds stylistically equivalent to an authentic Chopin nocturne. Cope admits EMI is not intelligent at all—in fact, he cares only about the impact of the music, rather than the origination of the compositions.

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[24] Horn.
[25] Horn, 8:29.

# III

## One Step Further:

## Mathematical Approaches to Analysis in the Twentieth Century and Beyond

There exists a historical dispute on the interrelation of music and mathematics from two ancient Greek philosophers, Pythagoras and Aristoxenus. Influenced by the thoughts of Boethius, Pythagoras (in addition to Plato) viewed music and mathematics in a relationship, along with astronomy and geometry, which he titled the *quadrivium.*[26] From Chapter I, it is known that Pythagoras' creation—and manipulation—of the monochord inspired thought of pitch relation on ratios mathematics. The *quadrivium* is just the foundation of Pythagorean thought and the *Harmony of the Spheres*. Aristoxenus, philosopher and creator of *Elements of Harmony*, thought differently of the relationship of math and music, and believed Pythagoras to overanalyze music with the mathematical study of pitch. In objection of Pythagorean thought, Aristoxenus proclaimed, "For just as it is not necessary for him who writes an Iambic to attend to the arithmetical proportions of the feet of which it is composed, so it is not necessary for him who writes a Phrygian song to attend to the ratios of the sounds proper thereto."[27] Aristoxenus strongly believed that "the nature of melody is best discovered by the perception of sense, and is retained by memory; and that there is no other way of arriving at the knowledge of music."[28] The thought of music as influenced by mathematics and

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26 Fauvel, J., R. Flood & R. Wilson (Eds.), 2003, p. 6.
27 Hawkins, 1868. pp. 66–67.
28 Ibid.

logic is ancient; however, the research, theory, and compositions spurred by these ideas are radical and innumerable.

Since the recognition of the interrelation of math and music, music theorists and mathematicians alike have created theories and methods by which to analyze music under a mathematical context. Two such theorists, David Lewin and Ernö Lendvai are the subjects of study in this chapter. In creating hybrids of advanced mathematics and theoretical analysis, these theorists have created innovative and complex lenses with which music can be analyzed. David Lewin is recognized for his research and application of abstract-algebraic set and group theory on pitch-class sets and intervals. In contrast, Ernö Lendvai's studies center themselves around the Fibonacci numbers, the Golden Ratio, and the music of Béla Bartók. The effectiveness of these styles of analysis is up for debate, but there is no questioning the impact these analyses have on the world of music and mathematics.

**David Lewin and Transformational Theory**

David Lewin studied mathematics as an undergraduate at Harvard University and continued his education in music theory and composition, earning his Master's at Princeton. There, he studied with Roger Sessions, Milton Babbitt, and Edward Cone, all of whom, without question, influenced his musical interests and analytical approach to music.[29] Aside from an impressive education resume, David Lewin soon became a household name in the 20th-century music theory circle with his introduction of set theory

[29] Cohn, 2003.

analysis in music, an idea initially created by theorist Allen Forte.[30] At the forefront of his research, Lewin's work centers on using set and group theory to understand the relations and transformations on intervals and chords.[31] Instead of observing notes on a staff, Lewin thinks of pitches as points on a pitch "plane" or space (similar to points on a coordinate plane). The intervals between these pitches are then the calculated distances between the points on the coordinate plane.

Before delving into Lewin's set theory of both math and music, one must first understand set and group theory of both music and math separately. Collins Dictionary of Mathematics defines a set as "a collection, possibly infinite, of distinct numbers, objects, etc. that is treated as an entity in its own right, and with identity dependent only upon its members" and set theory as "the elementary study of the properties of finite sets, or classes, and their relations."[32] As an example of a set, the dictionary gives the example of the set {3, the moon},[33] defining it to be the same as the set {the moon, 3} and {the only known natural earth satellite, the smallest odd prime number};[34] it is clear that order does not matter in a mathematically defined set. There is a similar sense of set equality in pitch-class sets, as observed in the following paragraph. In Herstein's *Topics in Algebra*, a group is defined as follows: "a nonempty set *G* is said to form a *group* if in *G* there is defined a binary operation… such that *G* has properties of closure, associativity, identity, and inverse."[35] The most important of these four properties is likely the inverse, as inversions are heavily present in music, especially in 12-tone row

[30] Forte, 1973.
[31] Cohn, 2003.
[32] Borowski & Borwein, 2007, pp. 509–510.
[33] In mathematics, brackets signify set notation, and commas separate elements of sets.
[34] Borowski & Borwein, 2007, p. 510.
[35] Herstein, 1975, p. 28.

matrix construction. Set theory in music deals heavily with pitch classes, or pitch class sets. A pitch class refers “to any one of the 12 notes of the equal-tempered scale regardless of spelling or octave placement.”[36] Pitch class sets hold properties of equality over octaves and enharmonics; for example, the pitch class of C contains all pitched C’s in music. A pitch class interval is simply “an integer between 0 and 11 that indicates the distance in half steps from the first pitch class up to the second pitch class.”[37] This leads to a final elementary explanation of the modular arithmetic in pitch class interval counting. Pitches are conceptualized using “modulo 12,” or mod 12 arithmetic. If we were to assign the number 0 to the pitch C, the C an octave higher would be assigned 12, which is equivalent to 0 (mod12). For a visual understanding, we look at the pitch class clock, as seen in Figure 8. We assign the numbers of a clock to the different pitch classes and use the knowledge of the musical scale that rises from A to G#, chromatically, and then returns to A. With these tools, it is simple to see that studying the 12 pitch classes is, in many ways, the same as observing hours on a 12-hour clock. In constructing matrices for 12-tone row serialist works, the numbers are still applied to pitches in a chromatic, ascending pattern; 0 is applied to the first pitch in the row, and the remaining 10 integers applied chromatically.

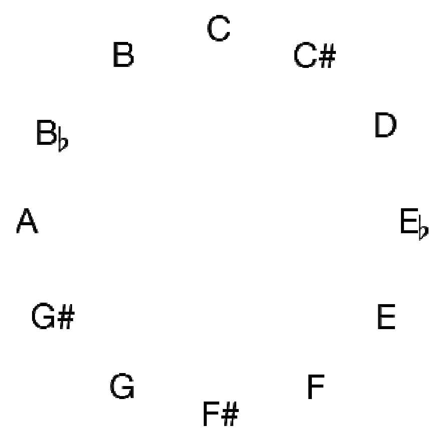


**Figure 8: The Pitch Class Clock**

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[36] Kostka, 2012, p. 2.
[37] Kostka, 2012, pp. 168–169.

David Lewin's transformational theory is an amalgamation of the set and group theories of both mathematics and music. In fact, it is the application of group theory to atonal and serial repertoire.[38] In Generalized Interval Systems, heretofore referred to as GIS, Lewin thinks of the transformations of pitch classes—transposition, retrograde, inversion, rhythmic shifts, chord inversion, etc.—as actions of group theory on the sets of pitch classes defined by the GIS. Lewin uses the properties of 12-tone row matrices, going a step further to critically analyze the transformations of the rows created by the directions of the matrix using ideas from mathematical set theory.

In order to understand Lewin's transformational theory, we will observe his definition of generalized interval systems:

> A *generalized interval system* consists of a set *S* of *musical elements*, a mathematical group *IVLS* which consists of the *intervals* of the generalized interval system, and an *interval function* int: S x S → IVLS such that:[39]
>
> - For all r, s, t ∈ S, we have int(r,s)·int(s,t) = int(r,t)
> - For every s ∈ S and every i ∈ IVLS, there exists a unique t ∈ S such that int(s,t) = i.

In short, for any three distinct pitches *r*, *s*, and *t* in the set *S*, the interval between *r* and *t* is equivalent to the sum of the intervals between *r* and *s*, and *s* and *t*. Additionally, for two pitches *s* and *t* in *S*, there is some interval, *i*, in the set *IVLS*, that is the interval between *s* and *t*. Though this axiomatic setting is quite intuitive and seems excessive, its purpose is

[38] Cohn, 2003.
[39] Fiore, 2009, p. 12.

The symbol ∈ may be read, "is an element of." The notation "int( " can be read as "interval." The · symbol means the composition of these two intervals—that is, the lengths of the two intervals are combined to determine the length of the third interval.

to define pitches and intervals in mathematical notation so as to begin building the foundation for the in-depth and extraordinary theory behind Lewin's GIS. Studying the measures of intervals between pitch classes, and assigning mathematical definitions to these musical concepts, provides a consistent way of labeling and organizing elements of music for further observation. Additionally, GIS provides ways to categorize, organize, and number transpositions and inversions in singular pieces and in collections of compositions from any composer.

**Ernö Lendvai, Béla Bartók, and the Fibonacci Numbers**

Ernö Lendvai is most well known for his studies on the music of Béla Bartók and the mathematical principles of the Fibonacci numbers and the Golden Ratio. Lendvai's work completely alters the analytical approach and thought around Béla Bartók's music. Instead of a traditional analysis, Lendvai offers a mathematical lens by which to observe number patterns in the notes, lengths, and chords of Bartók's music. We must first observe the Fibonacci numbers and the Golden Ratio to understand the analytical theory of Lendvai's work.

The Fibonacci numbers are, in list form, [0, 1, 1, 2, 3, 5, 8, 13, 21, 34,…, $a_{n-1}$, $a_n$, $a_{n+1}$], and are often represented as a sequence using the notation $a_{n+1} = a_n + a_{n-1}$. A number in the sequence is formed by the sum of the two previous numbers. The golden mean is formed as a result of this pattern, and is defined as the limit of the ratios of the successive terms in the sequence. The mathematical value of the golden mean is $^{(\sqrt{5} + 1)}/_{2}$, which is approximately 1.6180339887. Ratio values of the numbers in the Fibonacci series get closer and closer to the Golden Ratio as the values become larger and larger.

The Fibonacci elements $a_8$ and $a_9$ have ratio 21:34, which is equivalent to ~1:1.619047619, which in and of itself it less than a hundredth away from the golden mean. Elements $a_{19}$ and $a_{20}$ are even closer—their ratio is 4181:6765, which is equivalent to ~1:1.6180339632.

Béla Bartók's *Music for String, Percussion, and Celeste*, is perhaps the most studied in the application of the Golden Ratio to the harmony and structure of music. The following is an observation of Lendvai's application of the Golden Ration onto its musical structure.[40] In the fugue (first movement), the architectural element of dynamics takes on a form following the Fibonacci numbers, and further, the Golden Ratio. The piece makes dynamic shifts from pianissimo to forte-fortissimo (which appears near the midpoint of the piece), and decrescendos back to piano-pianissimo by the end of the piece. Though these shifts are not dynamically uncommon, it is the space between them that seems peculiar, and what spurs Lendvai's interest of the impact of the Golden Ratio. The dynamic motion divides the 89 bars of the fugue into two sections of 55 and 34 measures each. These are further divided into 6 sub-intervals through an observation of melodic motion, smaller dynamic contrasts, and the placement of performance elements throughout the fugue. Looking at the numeric values of the measure-lengths of each

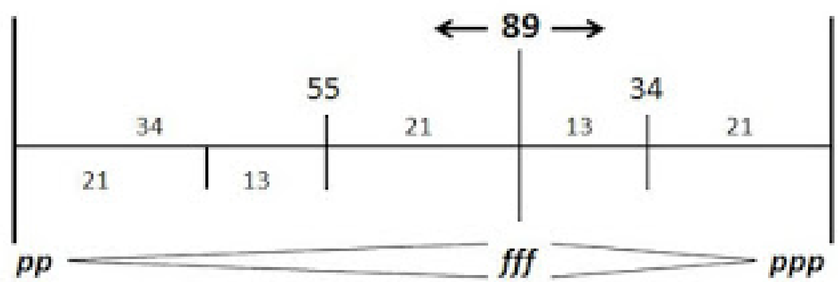


**Figure 9: Lendvai's diagram of the fugue from *Music for Strings, Percussion and Celesta***

[40] Lendvai, 1971, pp. 27–29.

division, it is clear there is a pattern. In the largest division, 55+34, the ratio created is ~1:1.617647—less than *one thousandth* away from the Golden Ratio. Stemming from here, Lendvai divides the 55-bar section further into 34+21 (~1:1.619048) and its 34-bar section into 13+21 (~1:1.625385), and finally, the 34-bar section of the initial 55-bar section into 21+13 (imitating the second part of the first division). This analysis continues, and Lendvai analyzes the entirety of Bartók's *Music for Strings, Percussion, and Celesta*. In this analysis, Lendvai finds numerous ways in which the composition mirrors the numbers of the Fibonacci sequence and creates ratios nearly equivalent to the Golden Ratio.

Though this analysis makes sense in the context of this work, in addition to many other Bartók pieces, there is question as to whether or not Lendvai's synthesis of the Golden Ratio into the music is forced. It is clear that the Fibonacci numbers are relevant in the compositions, as the numbers align themselves with the musical elements of the pieces almost flawlessly. Perhaps it was Bartók's intent to compose pieces following the Golden Ratio—it is difficult to ignore how well so many of his compositions align themselves harmonically, melodically, and stylistically with the Golden Ratio. Observing elements like dynamics, thematic motion, modulations, and others throughout Bartók's works makes it impossible to ignore the presence of the Golden Ratio. Though I do not discredit Lendvai of years of research and analysis, I am curious of the future of the Golden Ratio—and mathematics in general—in music. This curiosity, in addition to the research completed throughout the thesis process, leaves me with three final questions. First, what is the purpose of thinking of music in a mathematically structured way? Second, what can musicians take away from this style of analysis, especially of

Bartók, as the Golden Ratio demands attention in many Bartók compositions? Finally, what other composers have a mathematical pattern present throughout a multitude of their compositions?

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