2015

Geometric Limits of Julia Sets of Maps z^n + exp(2πiθ) as n → ∞

Scott R. Kaschner
Butler University, skaschne@butler.edu

Reaper Romero

David Simmons

GEOMETRIC LIMITS OF JULIA SETS WITH PARAMETERS ON THE CIRCLE

SCOTT R. KASCHNER, REAPER ROMERO, AND DAVID SIMMONS

ABSTRACT. We show that the geometric limit as $n \to \infty$ of the Julia sets $J(P_{n,c})$ for the maps $P_{n,c}(z) = z^n + c$ does not exist for almost every c on the unit circle. Furthermore, we show that there is always a subsequence along which the limit does exist and equals the unit circle.

Consider the family of maps

$$P_{n,c}(z) \;=\; z^n + c,$$

where $n \geq 2$ is an integer and $c \in \mathbb{C}$ is a parameter. These maps all share the quality that there is only one free critical point; that is, the critical point at infinity is fixed under iteration, and the iterates of the remaining critical point, $z = 0$, depend on both c and n. Because of this uni-critical property, many dynamical properties of the classical quadratic family $z \mapsto z^2 + c$ are also exhibited by this family of maps. Details of this family are readily available in the literature [6, 8, 5].

In this note, we will consider the filled Julia set $K(P_{n,c})$, the set of points in $\mathbb{C}$ that remain bounded under iteration and its boundary, the Julia set $J(P_{n,c})$. In [2], the structure of the filled Julia set $K(P_{n,c})$ and its boundary $J(P_{n,c})$, the Julia set, as $n \to \infty$ was examined. One of the major results is this work was

Theorem [Boyd-Schulz]. *Let $c \in \mathbb{C}$, and let $CS(\hat{\mathbb{C}})$ denote the collection of all compact subsets of $\hat{\mathbb{C}}$. Then under the Hausdorff metric $d_{\mathcal{H}}$ in $CS(\hat{\mathbb{C}})$,*

(1) *If $c \in \mathbb{C}\backslash\overline{\mathbb{D}}$, then*
$$\lim_{n\to\infty} J(P_{n,c}) = \lim_{n\to\infty} K(P_{n,c}) = S^1.$$
(2) *If $c \in \mathbb{D}$, then*
$$\lim_{n\to\infty} J(P_{n,c}) = S^1 \text{ and } \lim_{n\to\infty} K(P_{n,c}) = \overline{\mathbb{D}}.$$
(3) *If $c \in S^1$, then if $\lim_{n\to\infty} J(P_{n,c})$ and/or $\lim_{n\to\infty} K(P_{n,c})$ (and/or any liminf or limsup) exists, it is contained in $\overline{\mathbb{D}}$.*

The purpose of this note is to improve part (3) of this result. While there may be no limit as $n \to \infty$ for $J(P_{n,c})$ or $K(P_{n,c})$, experimentation suggests given $c \in S^1$, there is almost always a predictable pattern for the filled Julia set for $P_{n,c}$ as $n \to \infty$. This experimentation led to the following result:

Theorem 1. *Let $c = e^{2\pi i\theta} \in S^1$ such that $\theta \neq 0$ and $\theta \neq \frac{3q\pm1}{3(6p-1)}$ for any $p \in \mathbb{N}$ and $q \in \mathbb{Z}$. Then*

$$\lim_{n\to\infty} J(P_{n,c}) \text{ and } \lim_{n\to\infty} K(P_{n,c})$$

do not exist. Moreover, if θ is rational, $\theta \neq 0$, and $\theta \neq \frac{3q\pm1}{3(6p-1)}$, then there exist N and subsequences a_k and b_k partitioning $\{n \in \mathbb{N} \colon n \geq N\}$ such that

$$\lim_{k\to\infty} K(P_{a_k,c}) \;=\; S^1 \qquad \text{and} \qquad \lim_{k\to\infty} K(P_{b_k,c}) \;=\; \overline{\mathbb{D}}.$$

In Section 2, we present the background material and motivation for this result. The proof of Theorem 1 is the focus of Section 3.

The authors are grateful to Mikhail Stepenov at the University of Arizona for his helpful suggestions.

2. Background and Motivation

2.1. **Notation and Terminology.** The main results in this note rely on the convergence of sets in $\hat{\mathbb{C}}$, where the convergence is with respect to the Hausdorff metric. Given two sets A, B in a metric space (X, d), the Hausdorff distance $d_{\mathcal{H}}(A, B)$ between the sets is defined as

$$\begin{aligned} d_{\mathcal{H}}(A,B) &= \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\} \\ &= \max\left\{\sup_{a\in A}\inf_{b\in B} d(a,b), \sup_{b\in B}\inf_{a\in A} d(a,b)\right\}. \end{aligned}$$

Each point in A has a minimal distance to B, and vice versa. The Hausdorff distance is the maximum of all these distances. For example, a regular hexagon A inscribed in a circle B of radius r has sides of length r. In this case, $d_{\mathcal{A}}(A, B) = r(1 - \sqrt{3}/2)$, the shortest distance from the circle to the midpoint of a side of the hexagon. See Figure 3. Julia sets $J(P_{n,c})$ and filled Julia sets $K(P_{n,c})$ are compact [1] in the compact space $\hat{\mathbb{C}}$. Moreover, with the Hausdorff metric $d_{\mathcal{H}}$, $\hat{\mathbb{C}}$ is complete [3].

Suppose S_n and S are compact subsets of $\mathbb{C}$. If for all $\epsilon > 0$, there is $N > 0$ such that for any $n \geq N$, we have $d_{\mathcal{H}}(S_n, S) < \epsilon$, then we say S_n converges to S and write $\lim_{n\to\infty} S_n = S$.

We adopt the notation from [2]. For an open annulus with radii $0 < r < R$,

$$\mathbb{A}(r, R) := \{z \in \mathbb{C} \colon r < |z| < R\}.$$

Also, the open ball of radius $\epsilon > 0$ centered at z will be denoted $B(z, \epsilon)$.

2.2. **Motivation.** A basic fact from complex dynamics (see [1] or [7]) is that $K(P_{n,c})$ is connected if and only if the orbit of 0 stays bounded; otherwise it is totally disconnected. For each $n \geq 2$, we define the Multibrot sets

$$\mathcal{M}_n := \{c \in \mathbb{C} \colon J(P_{n,c}) \text{ is connected}\}.$$

Since 0 is the only free critical point, $\mathcal{M}_n$ is also the set of parameters c such that the orbit of 0 under iteration by $P_{n,c}$ remains bounded [7]. Since the maps $P_{n,c}$ are uncritical, much of their dynamical behavior mimics the family of complex quadratic polynomials [8].

It was proven in [2] that for sufficiently large N,

(1) $c \in \mathbb{D}$ implies for any $n \geq N$, $0 \in K(P_{n,c})$ (the orbit of 0 is bounded and $c \in \mathcal{M}_n$), and
(2) $c \in \mathbb{C}\backslash\overline{\mathbb{D}}$ implies for any $n \geq N$, $0 \notin K(P_{n,c})$ (the orbit of 0 is not bounded and $c \notin \mathcal{M}_n$).

For parameters $c \in S^1$, $P_{n,c}(0) \in S^1$ for any n, and this obstructs the direct proof that the orbit of 0 remains bounded (or not). However, one finds that in most cases, $P^2_{n,c}(0) \notin S^1$ and should expect that in these situations, determining whether the orbit of zero stays bounded depends heavily on where $P^2_{n,c}(0)$ is relative to the circle. In fact, working with the second iterate of 0 will be sufficient for all of our proofs.

Noting that $P^2_{n,c}(0) = P_{n,c}(c)$, we have the following convenient formula:

Proposition 1. *For $c = e^{2\pi i\theta} \in S^1$ and any positive integer n, $|P_{n,c}(c)| \geq 1$ if and only if*

$$\cos(2\pi\theta(n-1)) \geq -\frac{1}{2},$$

where equality holds if and only if $|P_{n,c}(c)| = 1$.

Proof. Note first that for $c = e^{2\pi i\theta}$, we have $P_{n,c}(c) = (e^{2\pi i\theta})^n + e^{2\pi i\theta}$, so

$$\begin{aligned} P_{n,c}(c) &= \cos(2\pi\theta n) + i\sin(2\pi\theta n) + \cos(2\pi\theta) + i\sin(2\pi\theta) \\ &= \cos(2\pi\theta n) + \cos(2\pi\theta) + i(\sin(2\pi\theta n) + \sin(2\pi\theta)). \end{aligned}$$

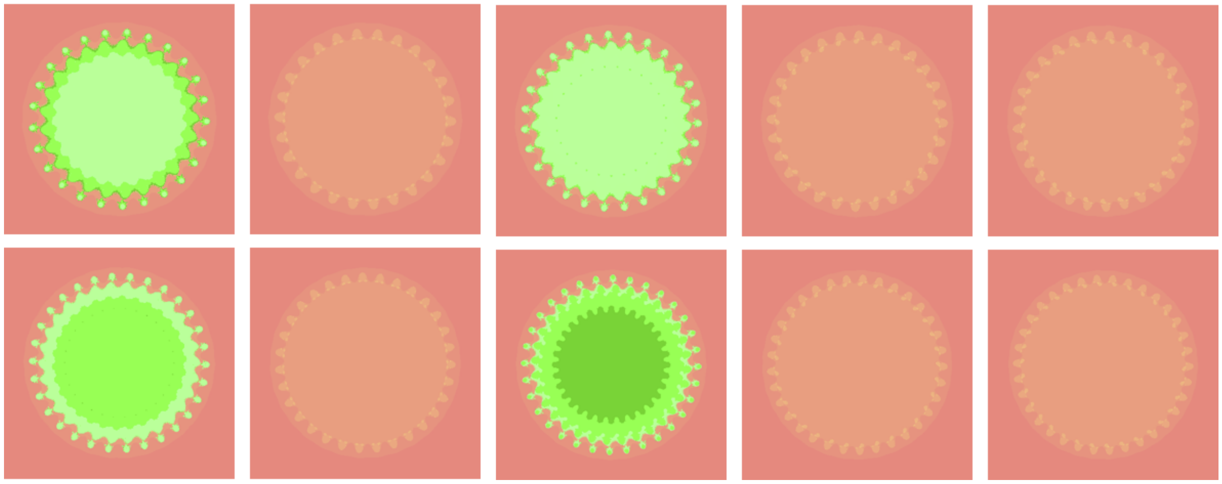

FIGURE 1. $J(P_{n,c})$ for $c = e^{4\pi i/5}$ and $n = 25 \dots 34$, starting from the upper left to the lower right.

If $P_{n,c}(c) \geq 1$, then

$$\begin{aligned} 1 &\leq (\cos(2\pi\theta n) + \cos(2\pi\theta))^2 + (\sin(2\pi\theta n) + \sin(2\pi\theta))^2 \\ &= 2\cos(2\pi\theta n)\cos(2\pi\theta) + 2\sin(2\pi\theta n)\sin(2\pi\theta) + 2 \\ &= 2\cos(2\pi\theta(n-1)) + 2 \end{aligned}$$

from which the result follows. □

Experimentation indicates that $P_{n,c}(c)$ being inside (or outside) S^1 very consistently dictates that $c \in \mathcal{M}_n$ (or $c \notin \mathcal{M}_n$). See Figure 1. Then the condition on $P_{n,c}(c)$ from Proposition 1 can be used to very consistently predict the structure of $K(P_{n,c})$, which Proposition 1 also suggests is periodic with respect to n. This will be made precise (with quantifiers) in Proposition 2 below.

More efficient experimentation with checking whether the orbit of 0 stays bounded clearly present this periodic (with respect to n) structure for $K(P_{n,c})$ when c is a rational angle on S^1. Figure 2 shows powers $421 \leq n \leq 450$ and $c = e^{\pi i p/q} \in S^1$ where $q = 15$ and p is an integer with $1 \leq p \leq 30$. A star indicates the Julia set $J(P_{n,c})$ is connected. There is, however, an inconsistency when the orbit of 0 remains on S^1. Note that the situation in which $P_{n,c}(c) \in S^1$ corresponds to having $\cos(2\pi\theta(n-1)) = -1/2$. This can be seen in Figure 2 for $n = 426$ and $2\theta = 26/15$ and $2\theta = 28/15$. The program that generated this data can provide a similar table for any equally distributed set of angles and any consecutive set of iterates.

This experimentation yields an intuition that is supported further by another result from [2]:

Theorem [Boyd-Schulz]. *Under the Hausdorff metric $d_{\mathcal{H}}$ in $CS(\hat{\mathbb{C}})$,*

$$\lim_{n\to\infty} M(P_{n,c}) = \overline{\mathbb{D}}.$$

For a fixed $c \in S^1$, as n increases, c will fall into and out of $\mathcal{M}_n$. See Figure 3. Thus, Proposition 1 provides nice visual evidence that this is truly periodic behavior. The Multibrot sets in Figure 3 are in logarithmic coordinates, so the horizontal axis is the real values $-1 \leq \theta \leq 1$, where $c = e^{2\pi i\theta}$. We are using logarithmic coords since we are interested in the angle θ.

It remains an open question what happens for parameters with angles $\theta = \frac{3q\pm1}{3(6p-1)}$ for $p \in \mathbb{N}$ and $q \in \mathbb{Z}$. We prove in Proposition 3 that the parameters corresponding to these angles force $P_{n,c}(c)$ to be a fixed point on S^1. In this case, the critical orbit is clearly bounded, so we know the filled

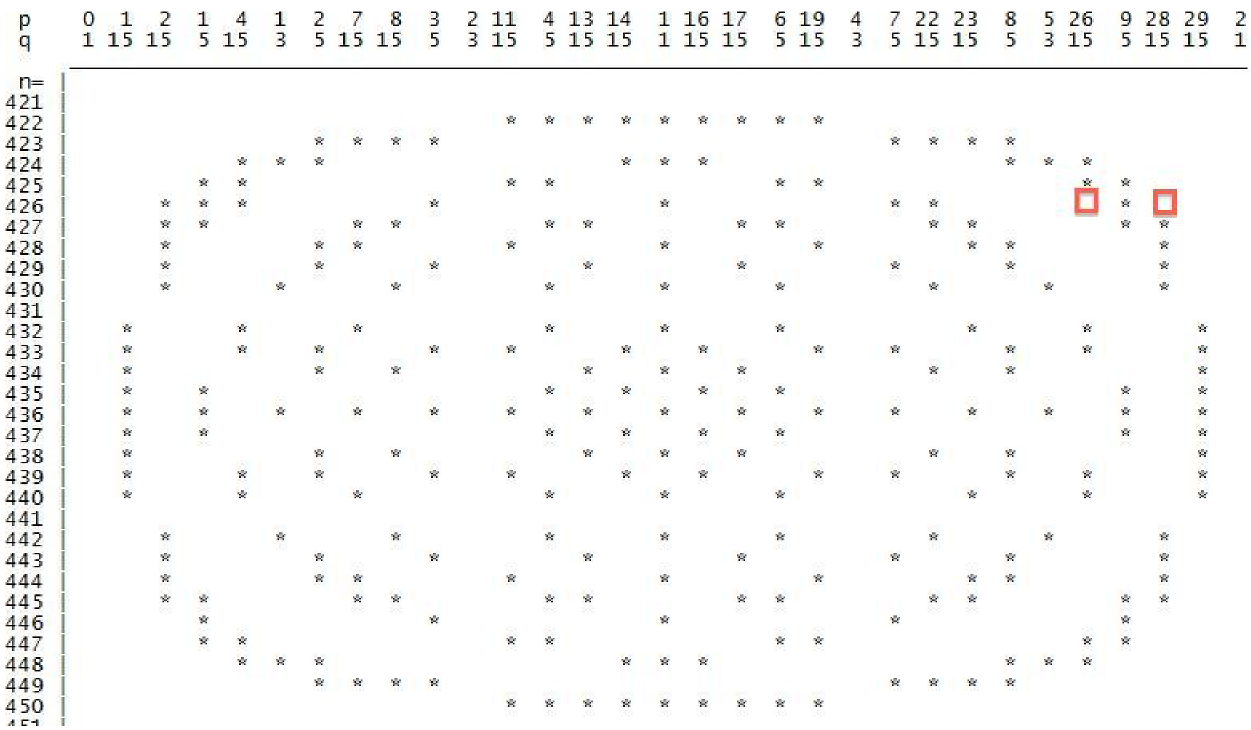

FIGURE 2. A star indicated $J(P_{n,c})$ is connected, where $c = e^{\pi i p/q}$

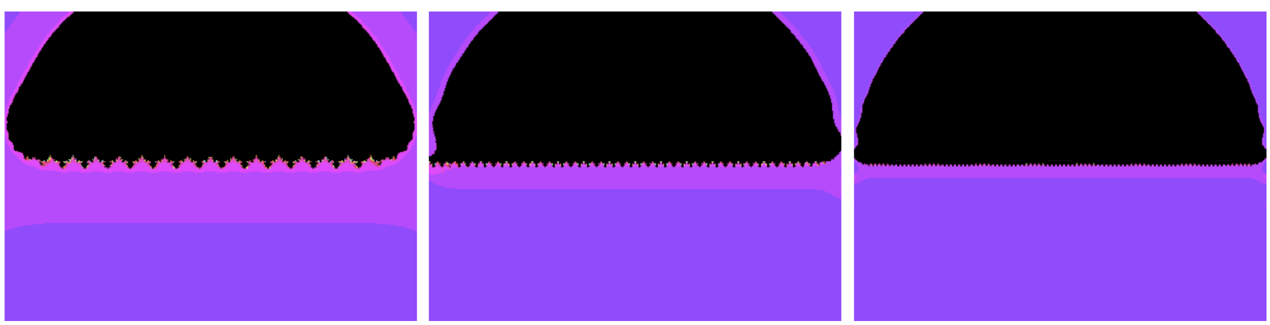

FIGURE 3. $\mathcal{M}_n$, where $c = e^{2\pi i\theta}$, $\theta \in \mathbb{C}$, and $n = 10, 25, 50$. Almost all fixed Reθ, falls into and out of $\mathcal{M}_n$ as n increases.

Julia set $K(P_{n,c})$ must be connected. See Figure 4. However, the behavior of the boundary $J(P_{n,c})$ is extremely complicated, as in the left-most image in Figure 4.

3. Proof of Theorem 1

We now prove that $P_{n,c}(c) \notin S^1$ does allow us to determine whether $c \in \mathcal{M}_n$.

Proposition 2. *Let $c \in S^1$. For any $\epsilon > 0$ there exists $N > 0$ so that for all $n \geq N$ one has:*

1. *if $|P_{n,c}(c)| < 1 - \epsilon$, then $\mathbb{D}_{1-\epsilon} \subset K(P_{n,c})$.*
2. *if $|P_{n,c}(c)| > 1 + \epsilon$, then $\mathbb{D}_{1-\epsilon} \subset \mathbb{C} \setminus K(P_{n,c})$.*

Noting that $0 \in \mathbb{D}_{1-\epsilon}$, it follows immediately from Propositions 1 and 2 that the orbit of 0 is bounded (or not) depending respectively on whether $P_{n,c}(c)$ is inside $\mathbb{D}_{1-\epsilon}$ (or outside $\mathbb{D}_{1+\epsilon}$). That is,

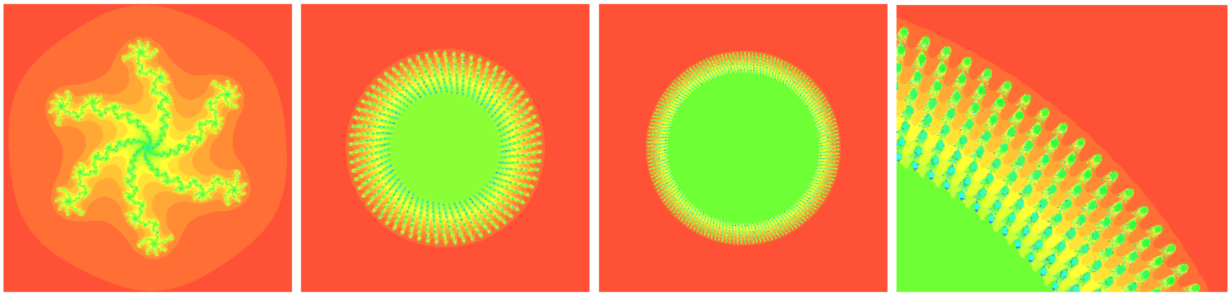

FIGURE 4. From left to right: $K(P_{n,c})$ for $c = e^{2\pi i/15}$ and $n = 6, 66, 156$. The far left image is a closer look at the boundary when $n = 165$

Corollary 1. *For all $\epsilon > 0$, there is an N such that for any $n \geq N$,*

1. *if $\cos(2\pi\theta(n-1)) < -1/2 - \epsilon/2$, then $K(P_{n,c})$ is connected and*
2. *if $\cos(2\pi\theta(n-1)) > -1/2+\epsilon/2$, then $K(P_{n,c})$ is totally disconnected and $K(P_{n,c}) = J(P_{n,c})$.*

Proof of Proposition 2. Fix $c \in S^1$. Let $\epsilon > 0$ and $r_n := |P^2_{n,c}(0)| = |c^n + c|$. Observe

$$\begin{aligned} \left|P^2_{n,c}(z)\right| \;=\; |(z^n+c)^n + c| \;&=\; \left|c^n + c + \sum_{k=1}^{n}\binom{n}{k}(z^n)^k c^{n-k}\right| \\ &\leq\; |c^n + c| + \sum_{k=1}^{n}\binom{n}{k}|z|^{nk} \;=\; r_n + (1+|z|^n)^n - 1. \end{aligned}$$

Then $|P^2_{n,c}(z)| \leq |z|$ when $r_n + (1+|z|^n)^n - 1 < |z|$. That is, for any $\eta \in (0,1)$, if

$$r_n \;\leq\; \eta + 1 - (1+\eta^n)^n, \tag{1}$$

then the disk $\mathbb{D}_\eta$ is forward invariant under $P^2_{n,c}$. Note that $(1+\eta^n)^n > 1$ and for fixed η, $(1+\eta^n)^n \to 1$ as $n \to \infty$. Fix $\eta = 1 - \epsilon/2$, so there is a positive integer N such that for all $n \geq N$,

$$(1+\eta^n)^n - 1 < \frac{\epsilon}{2}.$$

Thus, for any $n \geq N$ such that $r_n < 1 - \epsilon$,

$$r_n < \eta - \frac{\epsilon}{2} < \eta + 1 - (1+\eta^n)^n,$$

so, $\mathbb{D}_{1-\epsilon} \subset \mathbb{D}_\eta$ is forward invariant under $P^2_{n,c}$. This implies that the orbit of any point in $\mathbb{D}_{1-\epsilon}$ must be bounded in a disk of radius $\eta^n + 1$, so we have $\mathbb{D}_{1-\epsilon} \subset K(P_{n,c})$.

On the other hand, note that

$$\left|P^2_{n,c}(z)\right| \;=\; |(z^n+c)^n + c| \;\geq\; \left||c^n + c| - \sum_{k=1}^{n}\binom{n}{k}|z|^{nk}\right| \;=\; |r_n - (1+|z|^n)^n + 1|.$$

Again, fix $\eta = 1 - \epsilon/2$, so there is an N such that for any $n \geq N$, if $r_n > 1 + \epsilon$ and $|z| < 1 - \epsilon/2$, then

$$(1+|z|^n)^n - 1 < (1+\eta^n)^n - 1 < \frac{\epsilon}{2}.$$

That is, for $n \geq N$ and $z \in \mathbb{D}_\eta$,

$$|P^2_{n,c}(z)| \geq |r_n - (1+|z|^n)^n + 1| \geq 1 + \frac{\epsilon}{2}.$$

By Lemma 1, we can also choose N large enough that $K(P_{n,c}) \subset \mathbb{D}_{1+\epsilon/2}$ as well. Then for any $n > N$ and $z \in \mathbb{D}_\eta$, if $|P_{n,c}(c)| = r_n < 1 + \epsilon$, then $P^2_{n,c}(z) \notin K(P_{n,c})$. It follows that $z \notin K(P_{n,c})$, so $\mathbb{D}_\eta \subset \mathbb{C}\backslash K(P_{n,c})$. ◻

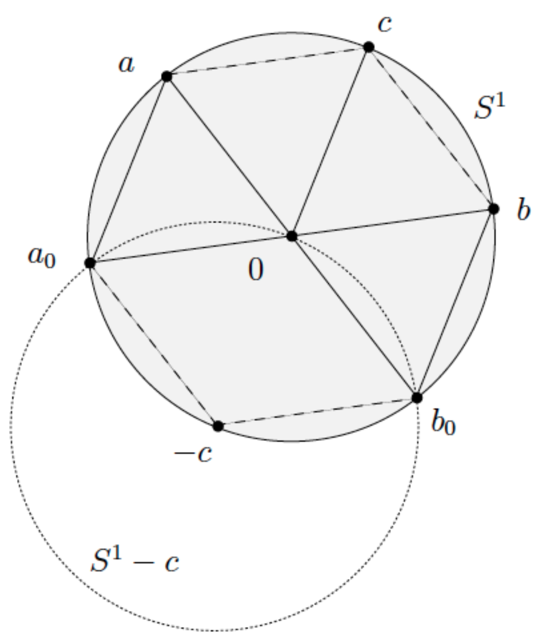

FIGURE 5. $P_{n,c}(c)$ is on the circle if and only if $c^n = a_0$ or $a^n = b_0$.

What remains is to examine $c \in S^1$ such that $P_{n,c}(c) \in S^1$ as well. This case is simpler and occurs less frequently than one might expect.

Proposition 3. *Let $c = e^{2\pi i\theta}$ and $P_{n,c}(z) = z^n + c$. Then $P^2_{n,c}(c) \in S^1$ if and only if $P_{nc}(c)$ is a fixed point, in which case, $(n, \theta) \in N$, where*

$$N := \left\{ (n,\theta) \in \mathbb{N} \times \mathbb{R} \mid n = 6p, \theta = \frac{3q \pm 1}{3(6p-1)}, \text{ where } p \in \mathbb{N} \text{ and } q \in \mathbb{Z} \right\}.$$

Proof. Since $|c| = 1$, note that the set $S^1 - c := \{z - c \mid z \in S^1\}$ is a circle centered at $-c \in S$, so it intersects S^1 in exactly two points, call them a_0 and b_0. By construction, $a_0 + c, b_0 + c \in S^1$, so define

$$\begin{aligned} a &:= a_0 + c \\ b &:= b_0 + c. \end{aligned}$$

Moreover, the points $\{c, a, a_0, -c, b_0, b\}$ form a hexagon inscribed in S^1 whose sides are all length one. Thus, we have

$$\begin{aligned} a &= e^{2\pi i(\theta+1/6)} \\ a_0 &= e^{2\pi i(\theta+1/3)} \\ b_0 &= e^{2\pi i(\theta-1/3)} \\ b &= e^{2\pi i(\theta-1/6)}. \end{aligned}$$

See Figure 3. For any $z \in S^1$, we have that $P_{n,c}(z) = z^n + c$ and $z^n \in S^1$, so $P_{n,c}(z) \in S^1$ if and only if

$$z^n \in (S^1 - c) \cap S^1 = \{a_0, b_0\};$$

that is, $P_{n,c}(z) \in \{a, b\}$. It follows that $|P^k_{n,c}(c)| = 1$ for all $k \geq 0$ if and only if one of the following is true: a is a fixed point, b is a fixed point, or a and b are a two-cycle.

Assume that $P_{n,c}(c) \in S^1$. First observe that $P_{n,c}(c) \in \{a, b\}$, so

$$P_{n,c}(c) = e^{2\pi i(\theta \pm 1/6)}.$$

Since $P_{n,c}(c) = c^n + c = e^{2\pi i\theta n} + e^{2\pi i\theta}$, it follows that

$$e^{2\pi i\theta n} = e^{2\pi i(\theta \pm 1/6)} - e^{2\pi i\theta} = e^{2\pi i(\theta \pm 1/3)}.$$

Thus, $\theta n = \theta \pm 1/3 + q$ for some integer q, so

$$\theta(n-1) = q + \frac{1}{3} \text{ if } P_{n,c}(c) = a \text{ and} \tag{2}$$

$$\theta(n-1) = q - \frac{1}{3} \text{ if } P_{n,c}(c) = b. \tag{3}$$

Proceeding to the next iterate, note that $P^2_{n,c}(c) \in \{a, b\}$ as well, so we need only examine $P_{n,c}(a)$ and $P_{n,c}(b)$. Since $P_{n,c}(a), P_{n,c}(b) \in \{a, b\}$, it must be for some integer p_0,

$$P_{n,c}\left(e^{2\pi i(\theta \pm 1/6)}\right) = e^{2\pi i(\theta \pm 1/6)n} + e^{2\pi i\theta} \in \{a, b\} = \left\{e^{2\pi i(\theta + 1/6 + p_0)}, e^{2\pi i(\theta - 1/6 + p_0)}\right\}.$$

Then it follows that from the definition of a and b that $e^{2\pi i(\theta \pm 1/6 + p_0)} \in \{a_0, b_0\}$, so we have $(\theta \pm 1/6)n = \theta \pm 1/3 + p_0$. In particular,

$$(n-1)\theta = p_0 + \frac{1}{3} - \frac{n}{6}, \quad \text{if } P_{n,c}(a) = a, \tag{4}$$

$$(n-1)\theta = p_0 - \frac{1}{3} - \frac{n}{6}, \quad \text{if } P_{n,c}(a) = b, \tag{5}$$

$$(n-1)\theta = p_0 + \frac{1}{3} + \frac{n}{6}, \quad \text{if } P_{n,c}(b) = a, \text{ and} \tag{6}$$

$$(n-1)\theta = p_0 - \frac{1}{3} + \frac{n}{6}, \quad \text{if } P_{n,c}(b) = b. \tag{7}$$

If a and b are a two cycle, then equations (5) and (6) together imply $q \pm 1/3 = p_0$. This contradicts the fact that q and p_0 are both integers. A similar contradiction arises from the cases when $P_{n,c}(b) = a$ and a is fixed, or when $P_{n,c}(a) = b$ and b is fixed.

The only remaining possibilities are that $P_{n,c}(c) = P_{n.c}(a) = a$ or $P_{n,c}(c) = P_{n.c}(b) = b$. Thus, we have shown that $|P^k_{n,c}(c)| = 1$ for all $k \geq 0$ if and only if for all $k \geq 1$, $P^k_{n,c}(c) = a$ or $P^k_{n,c}(c) = b$.

It remains to show that $(n, \theta) \in N$ is an equivalent statement. Supposing that for all $k \geq 1$, $P^k_{n,c}(c) = a$ or $P^k_{n,c}(c) = b$, we have

$$q \pm \frac{1}{3} = \theta(n-1) = p_0 \pm \frac{1}{3} \mp \frac{n}{6}.$$

From this equation, one can see that $n = 6p$, where $p = q - p_0 \in \mathbb{N}$. Moreover, the equations (2) and (3) derived from the first iterate of c yield

$$\theta(n-1) = q \pm \frac{1}{3},$$

so

$$\theta = \frac{3q \pm 1}{3(n-1)} = \frac{3q \pm 1}{3(6p-1)}.$$

□

The following lemmas are from [2]. The third is a subtle variation, so we include the proof.

Lemma 1 (Boyd-Schulz)**.** *Let $c \in \mathbb{C}$. For any $\epsilon > 0$, there is an N such that for all $n \geq N$,*

$$K(P_{c,n}) \subset \mathbb{D}_{1+\epsilon}.$$

Lemma 2 (Boyd-Schulz)**.** *Let $z \in J(P_{n,c})$. If ω is an n-th root of unity, then $\omega z \in J(P_{n,c})$.*

Lemma 3 (Boyd-Schulz). *Let $\epsilon > 0$ and $c = e^{2\pi i\theta} \in S^1$ such that $\theta \neq \frac{3q\pm1}{3(6p-1)}$ for any $p \in \mathbb{N}$ and $q \in \mathbb{Z}$. There is an $N \geq 2$ such that for all $n \geq N$ and for any $e^{i\phi} \in S^1$,*

$$B(e^{i\phi}, \epsilon) \cap J(P_{n,c}) \neq \emptyset.$$

Proof. By Proposition 2, there is an N_1 such that for any $n \geq N_1$, we have $J(P_{n,c}) \subset \mathbb{A}(1-\epsilon/2, 1+\epsilon/2)$. Let $e^{i\phi} \in S^1$ and $\alpha > 0$ be the angle so that

$$U := \{re^{i\tau} \colon r > 0, \phi - \alpha < \tau < \phi + \alpha\} \cap \mathbb{A}(1-\epsilon/2, 1+\epsilon/2\}$$

is contained in $B(e^{i\phi}, \epsilon)$. The same α works for each different ϕ.

For any n, let $\omega_n = e^{2\pi i/n}$, and choose $N > N_1$ such that $2\pi/N < \alpha$, noting that N is also independent of ϕ. We have $2\pi/n < \alpha$ for any $n \geq N$.

Since $J(P_{n,c})$ is nonempty for any n [7], choose $z_n \in J(P_{n,c})$ for each $n \geq N$. Then for some integer $1 \leq j_n \leq n-1$, we have

$$\omega_n^{j_n} z_n \in U \subset B(e^{i\phi}, \epsilon).$$

Thus, for all $n \geq N$, $B(e^{i\phi}, \epsilon) \cap J(P_{n,c}) \neq \emptyset$. □

Proof of Theorem 1. Fix $c = e^{2\pi i\theta} \in S^1$ and assume $\theta \neq \frac{3q\pm1}{3(6p-1)}$ for any $p \in \mathbb{N}$ and $q \in \mathbb{Z}$. Then by Proposition 3, $|P_{n,c}(c)| \neq 1$, and by Proposition 1, we have $\cos(2\pi\theta(n-1)) \neq -\frac{1}{2}$. In particular,

(1) $|P_{n,c}(c)| < 1$ when $\cos(2\pi\theta(n-1)) < -\frac{1}{2}$, and
(2) $|P_{n,c}(c)| > 1$ when $\cos(2\pi\theta(n-1)) > -\frac{1}{2}$.

Note that $\cos(2\pi\theta(n-1))$ has period $1/\theta$ as a function of n. If θ is a rational number, then this function takes a finite number of values. In this case, $|P_{n,c}(c)|$ can be bound away from S^1 by a fixed distance for any n. Let $\epsilon > 0$ be smaller than this minimum distance. Then, Proposition 2 gives that that there is $N > 0$ such that for all $n \geq N$, we have either

1. $|P_{n,c}(c)| < 1 - \epsilon$ and $\mathbb{D}_{1-\epsilon} \subset K(P_{n,c})$, or
2. $|P_{n,c}(c)| > 1 + \epsilon$ and $\mathbb{D}_{1-\epsilon} \subset \mathbb{C} \setminus K(P_{n,c})$.

Moreover, if we consider θ as a rational rotation of the circle, the periodic orbit (with respect to n) induces intervals on S^1 that are permuted by this rotation [4]. Since $\cos(2\pi\theta(n-1)) \neq -\frac{1}{2}$, we must have n and m such that $\cos(2\pi\theta(n-1)) \geq -\frac{1}{2}$ and $\cos(2\pi\theta(m-1)) \geq -\frac{1}{2}$. Again, since this rotation is periodic, we can find such n and m for any $N > 0$. Thus, no limit as $n \to \infty$ can exist for $K(P_{n,c})$.

Now suppose θ is irrational. For any sufficiently small $\epsilon > 0$ let $N > 0$ be given by Corollay 1. Since the values $\cos(2\pi(n-1)\theta)$ are equidistributed in $[-1, 1]$ according to $\cos_*(\text{Leb})$ (where Leb is the Lebesgue measure on the circle) [4], there will be arbitrarily large values of $m, n > N$ such that $\cos(2\pi(n-1)\theta) < -1/2 - \epsilon$ and $\cos(2\pi(m-1)\theta) > -1/2 + \epsilon$. In this case $K_{n,c}$ contains the disc $\mathbb{D}_{1-\epsilon}$ while, $\mathbb{D}_{1-\epsilon}$ is contained in the complement of $K_{m,c}$. Thus, no limit as $n \to \infty$ can exist for $K(P_{n,c})$.

Having established the claim in Theorem 1 that no limit exists, we move on to prove the claim that if θ is rational, $\theta \neq 0$, and $\theta \neq \frac{3q\pm1}{3(6p-1)}$, then there are subsequences a_k and b_k partitioning $\{n \in \mathbb{N} \colon n \geq N\}$ such that

$$\lim_{k\to\infty} K(P_{a_k,c}) \;=\; S^1 \qquad \text{and} \qquad \lim_{k\to\infty} K(P_{b_k,c}) \;=\; \overline{\mathbb{D}}.$$

We know from Proposition 3 that $|P_{n,c}(c)| \neq 1$ for any positive integer n. Thus, for any $\epsilon > 0$, we can use Proposition 2 to find an $N \in \mathbb{N}$ and construct subsequences

$$\begin{aligned} A_\epsilon &= \{n \in \mathbb{Z}_+ \colon |P_{n,c}(c)| < 1 - \epsilon\} \text{ and} \\ B_\epsilon &= \{n \in \mathbb{Z}_+ \colon |P_{n,c}(c)| > 1 + \epsilon\} \end{aligned}$$

such that for any $n \geq N$,

(1) if $n \in A_\epsilon$, then $K(P_{n,c})$ is full and connected, and
(2) if $n \in B_\epsilon$, then $K(P_{n,c}) = J(P_{nc})$ is totally disconnected.

Moreover, as $\epsilon \to 0$, these two sets partition $\mathbb{N}$.

With the structure of $K(P_{n,c})$ consistent in each of the sets A_ϵ and B_ϵ, the remainder of the proof very closely follows the proof of Theorem 1.2 in [2].

Let $\epsilon > 0$ and a_k the subsequence of $n \in A_\epsilon$. Then $|P_{a_k,c}(c)| < 1 - \epsilon$, so by Proposition 1, there is an N_1 such that for any $a_k \geq N_1$, we have $\mathbb{D}_{1-\epsilon} \subseteq K(P_{a_k,c})$. By Lemma 1, there is an $N_2 \geq N_1$ such that for any $a_k \geq N_2$, we have $K(P_{a_k,c}) \subseteq \mathbb{D}_{1+\epsilon}$. Thus, for any $z \in K(P_{a_k,c})$,

$$d(z, \overline{\mathbb{D}}) = \inf_{w \in \overline{\mathbb{D}}} |z - w| < \epsilon.$$

Now let $w \in \overline{\mathbb{D}}$. Since $\mathbb{D}_{1-\epsilon} \subseteq K(P_{a_k,c}) \subseteq \mathbb{D}_{1+\epsilon}$, we have

$$d(w, K(P_{a_k,c})) = \inf_{z \in K(P_{a_k,c})} |z - w| < \epsilon.$$

If follows that

$$d_{\mathcal{H}}(K(P_{a_k,c}), \overline{\mathbb{D}}) \quad = \quad \max \left\{ \sup_{z \in K(P_{a_k,c})} d(z, \overline{\mathbb{D}}), \sup_{w \in \overline{\mathbb{D}}} d(w, K(P_{a_k,c})) \right\} < \epsilon.$$

Thus, $\lim_{k\to\infty} K(P_{a_k,c}) = \overline{\mathbb{D}}$.

Now let b_k be the subsequence of $n \in B_\epsilon$. Again, by Proposition 1 and Lemma 1, there is an N_1 such that for any $b_k \geq N_1$, we have $K(P_{n,c}) \subset \mathbb{A}(1 - \epsilon/2, 1 + \epsilon/2)$. Also, note that $0 \notin K(P_{n,c})$, so $K(P_{nc})$ is totally disconnected and $J(P_{n,c}) = K(P_{n,c})$. Then for any $z \in J(P_{b_k,c})$, we have

$$d(z, S^1) = \inf_{s \in S^1} |z - s| < \epsilon.$$

By Lemma 3, there is an $N_2 \geq N_1$ such that for any $b_k \geq N_2$ and for any $s \in S^1$,

$$d(s, J(P_{b_k,c})) = \inf_{z \in J(P_{b_k,c})} |z - s| < \epsilon.$$

Thus, it follows that $d_{\mathcal{H}}(J(P_{b_k,c}), S^1) < \epsilon$ and $\lim_{k\to\infty} J(P_{b_k,c}) = S^1$. □

References

[1] Alan F. Beardon. *Iteration of rational functions*, volume 132 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. Complex analytic dynamical systems.

[2] Suzanne Hruska Boyd and Michael J. Schulz. Geometric limits of Mandelbrot and Julia sets under degree growth. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 22(12):1250301, 21, 2012.

[3] Jeff Henrikson. Completeness and total boundedness of the hausdorff metric. *MIT Undergraduate Journal of Mathematics*, 1:69–80, 1999.

[4] Anatole Katok and Boris Hasselblatt. Introduction to the modern theory of dynamical systems. *Cambridge, Cambridge*, 1995.

[5] Mikhail Lyubich. Dynamics of quadratic polynomials. I, II. *Acta Math.*, 178(2):185–247, 247–297, 1997.

[6] Curtis T. McMullen. *Complex dynamics and renormalization*, volume 135 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1994.

[7] John Milnor. *Dynamics in one complex variable*, volume 160 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, third edition, 2006.

[8] Dierk Schleicher. On fibers and local connectivity of Mandelbrot and Multibrot sets. In *Fractal geometry and applications: a jubilee of Benoît Mandelbrot. Part 1*, volume 72 of *Proc. Sympos. Pure Math.*, pages 477–517. Amer. Math. Soc., Providence, RI, 2004.

E-mail address: skaschner@math.arizona.edu

University of Arizona Department of Mathematics, Mathematics Building, Room 317, 617 N. Santa Rita Ave., Tucson, AZ 85721-0089, United States

E-mail address: rromero@email.arizona.edu

University of Arizona Department of Mathematics, Mathematics Building, 617 N. Santa Rita Ave., Tucson, AZ 85721-0089, United States

E-mail address: davidsimmons@email.arizona.edu

University of Arizona Department of Mathematics, Mathematics Building, 617 N. Santa Rita Ave., Tucson, AZ 85721-0089, United States