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Some Curious Cut-Ups

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We have noticed a certain kind of \( n \)-gon dissection into triangles that has a wonderful property of interest to most puzzlists. Namely that any two triangles have at least one edge in common yet no two triangles need be congruent. In an informal poll of specialists at a recent convention, none of them saw immediately how this could be accomplished. But in fact it is very straightforward.

**Theorem:** Any cyclical \( n \)-gon whose circumcenter is contained in its interior can be dissected into \( 4n \) triangles each of which has a common side.

The proof of this easy theorem will become apparent after a few examples. But first recall some definitions. A cyclical \( n \)-gon is one whose vertices lie on a circle and the center of this circle is the circumcenter. Every triangle is cyclical but for a flattish triangle the circumcenter may be outside the triangle. Every regular \( n \)-gon is cyclical and contains its circumcenter. There are infinitely many irregular cyclical \( n \)-gons that contain their circumcenters too. All of these we will dissect.

Figure 1 shows two 3-gons (triangles), one equilateral and one scalene. \( O \) is the circumcenter in each so that \( OA, OB \) and \( OC \) are equal radii. The triangle points \( P_1, P_2 \) and \( P_3 \) are midpoints of the radii. Now choose points \( Q_1, Q_2 \) and \( Q_3 \) on the side of each triangle and connect the points as indicated. One obtains \( 4 \cdot 3 = 12 \) triangles each with one side a half-radius. If the \( Q \)-points are chosen judiciously no two of the twelve triangles need be congruent. This procedure can easily be generalized to any suitable \( n \)-gon yielding \( 4n \) triangles.

Of special interest are the number of solutions to the puzzle of reconstructing the \( n \)-gon from the pieces. Note that in Figure 1, the equilateral triangle has three congruent triangular parts that could be interchanged: \( \triangle AOB, \triangle AOC \) and \( \triangle BOC \). Not counting rotations or reflections as different this can be done in two ways. In the general regular \( n \)-gon case this leads to \((n-1)!\) different solutions (depending again on the choice of the \( Q \)-s). For the irregular scalene triangle note that there is only one solution.

The fact that any two pieces of any one puzzle fit along at least one side gives a bit of misdirection to the solver. Two-coloring the pieces could be a hint to help the solver.

One might suspect that inscribed circles could also be used to obtain interesting dissecting puzzles. Figure 4 shows one possibility. \( O \) is the incenter of \( \triangle ABC \) and the radii are halved. Notice there are always several sets of congruent triangles here that make the corresponding put-together puzzle, in our opinion, much too easy.
FIGURE 1
We are not adverse to choosing the Qs so that some triangular pieces are congruent. Figure 2 shows an equilateral triangle three-colored with four sets of congruent subtriangles. The Qs on the sides divide a side in the ratio 3:2. The coloring allows a solid color triangle to be formed using each color. Confront a potential solver with the 3 triangular parts and ask for a single triangle with no like colors abutting along a side.
A nice puzzle on a square can be constructed by the reader thusly. Take a square piece of paper and fold on the diagonals to locate point O. Fold in half twice horizontally to identify points P1, P2, P3 and P4. Smoothly paste the square on a piece of cardboard. Select randomly the points Q1, Q2, Q3, and Q4 on the four edges of the square. If these are not placed symmetrically the puzzle will be harder. Connect the thirteen points as shown and cut out the $4 \times 4 = 16$ triangles with a hobby knife. Color the parts if you like. This puzzle could have $(4-1)! = 6$ solutions.
FIGURE 4

\( \triangle ABC \) dissected with respect to its inscribed circle with incenter \( O \).
fig A
fig D