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Degree and neighborhood conditions for hamiltonicity of claw-free graphs

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Abstract

For a graph H, let $\sigma_t(H) = \min\{\Sigma_{i=1}^{t} d_H(v_i) \mid \{v_1, v_2, \cdots, v_t\}$ is an independent set in $H\}$ and let $U_t(H) = \min\{|\bigcup_{i=1}^{t} N_H(v_i)| \mid \{v_1, v_2, \cdots, v_t\}$ is an independent set in $H\}$. We show that for a given number ϵ and given integers $p \geq t > 0$, $k \in \{2, 3\}$ and $N = N(p, \epsilon)$, if H is a k-connected claw-free graph of order $n > N$ with $\delta(H) \geq 3$ and its Ryjáček's closure $cl(H) = L(G)$, and if $d_t(H) \geq t(n + \epsilon)/p$ where $d_t(H) \in \{\sigma_t(H), U_t(H)\}$, then either H is Hamiltonian or G, the preimage of $L(G)$, can be contracted to a k-edge-connected K_3-free graph of order at most $\max\{4p-5, 2p+1\}$ and without spanning closed trails. As applications, we prove the following for such graphs H of order n with n sufficiently large:

(i) If $k = 2$, $\delta(H) \geq 3$, and for a given t $(1 \leq t \leq 4)$ $d_t(H) \geq \frac{tn}{4}$, then either H is Hamiltonian or $cl(H) = L(G)$ where G is a graph obtained from $K_{2,3}$ by replacing each of the degree 2 vertices by a $K_{1,s}$ $(s \geq 1)$. When $t = 4$ and $d_t(H) = \sigma_4(H)$, this proves a conjecture in [15].

(ii) If $k = 3$, $\delta(H) \geq 24$, and for a given t $(1 \leq t \leq 10)$ $d_t(H) > \frac{t(n+5)}{10}$, then H is Hamiltonian. These bounds on $d_t(H)$ in (i) and (ii) are sharp. It unifies and improves several prior results on conditions involved σ_t and U_t for the hamiltonicity of claw-free graphs. Since the number of graphs of orders at most $\max\{4p - 5, 2p + 1\}$ are fixed for given p, improvements to (i) or (ii) by increasing the value of p are possible with the help of a computer.

Keywords: Claw-free graph, Hamiltonicity, Neighborhood condition, degree condition

1 Introduction

We shall use the notation of Bondy and Murty [2], except when otherwise stated. Graphs considered in this paper are finite and loopless. A graph is called a multigraph if it contains multiple edges. A graph without multiple edges is called a simple graph or simply a graph. As in [2], $\kappa'(G)$ and $d_G(v)$ denote the edge-connectivity of G and the degree of a vertex v in G, respectively. For a vertex $v \in V(G)$, let $E_G(v)$ be the set of edges incident with v in G. Then $d_G(v) = |E_G(v)|$. Define $\overline{\sigma}_2(G) = \min\{d_G(u)+d_G(v) \mid$ for every edge $uv \in E(G)\}$ and $D_i(G) = \{v \in V(G) \mid d_G(v) = i\}$. An edge cut X of a graph G is *essential* if each component of $G-X$ has some edges. A graph G is *essentially k-edge-connected* if G is connected and does not have an essential edge cut of size less than k. An edge $e = uv$ is called a *pendant edge* if $\min\{d_G(u), d_G(v)\} = 1$. The *independence number* of a graph G is denoted by $\alpha(G)$ and the *clique covering number of* G, (i.e. the minimum number of cliques necessary for covering $V(G)$) by $\theta(G)$. An independent set with t vertices is called a *t-independent set* and a matching with t edges is called a *t-matching*. A graph H is *claw-free* if H does not contain an induced subgraph isomorphic to $K_{1,3}$. A connected graph Ψ is a *closed trail* if the degree of each vertex in Ψ is even. A closed trail Ψ is called a spanning closed trail (SCT) in G if $V(G) = V(\Psi)$, and is called a *dominating closed trail* (DCT) if $E(G - V(\Psi)) = \emptyset$. A graph is *supereulerian* if it contains an SCT. The family of supereulerian graphs is denoted by $\mathcal{SL}$. A graph is Hamiltonian if it has a spanning cycle. Throughout this paper, we use P for the Petersen graph.

The line graph of a graph G is denoted by $L(G)$. A vertex $v \in V(H)$ is *locally conntected* if its neighborhood $N_H(v)$ induces a connected graph. The closure of a claw-free graph H introduced by Ryjáček [25] is the graph obtained by recursively adding edges to join two nonadjacent vertices in the neighborhood of any locally connected vertex of H as long as this is possible and is denoted by $cl(H)$. A claw-free graph H is said to be *closed* if $H = cl(H)$. The following theorem shows the relationship between a DCT of a graph and a Hamiltonian cycle in its line graph.

Theorem 1.1. *(Harary and Nash-Willams [16]). The line graph $H = L(G)$ of a graph G with at least three edges is Hamiltonian if and only if G has a DCT.*

Now, we define two families of nonhamiltonian claw-free graphs.

For a $K_{2,3}$, let $D_2(K_{2,3}) = \{v_1, v_2, v_3\}$. Let $\mathcal{K}_{2,3}(s_1, s_2, s_3, n)$ be the family of graphs of size n obtained from a $K_{2,3}$ by adding $s_i \geq 1$ pendant edges at v_i ($i = 1, 2, 3$) and $s_1 + s_2 + s_3 + 6 = n$.

Let $\mathcal{Q}_{2,3}(s_1, s_2, s_3, n) = \{H : H = L(G)$ where $G \in \mathcal{K}_{2,3}(s_1, s_2, s_3, n)\}$.

For the Petersen graph P, let $V(P) = \{v_1, \cdots, v_{10}\}$. Let $\mathcal{P}(n, s)$ be the family of graphs of size n obtained from P by replacing each v_i by a connected subgraph Φ_i with size $s_i \geq s$ and $15 + \sum_{i=1}^{10} s_i = n$. Let $\mathcal{P}_1(n, s)$ be the sub-family of $\mathcal{P}(n, s)$ in which each $\Phi_i = K_{1,s_i}$.

Let $\mathcal{Q}_P(n, s) = \{H : H = L(G)$, where $G \in \mathcal{P}(n, s)\}$.

Let $\mathcal{Q}_P^1(n, s) = \{H : H = L(G)$, where $G \in \mathcal{P}_1(n, s)\}$, a subfamily of $\mathcal{Q}_P(n, s)$.

By Theorem 1.1, graphs in $\mathcal{Q}_{2,3}(s_1, s_2, s_3, n) \cup \mathcal{Q}_P(n, s)$ are nonhamiltonian.

For a graph H and $t \geq 1$, we define

- $\sigma_t(H) = \min\{\Sigma_{i=1}^t d_H(v_i) \mid \{v_1, v_2, \cdots, v_t\}$ is an independent set in $H\}$ (if $t > \alpha(H)$, $\sigma_t(H) = \infty$);
- $U_t(H) = \min\{|\bigcup_{i=1}^t N_H(v_i)| \mid \{v_1, v_2, \cdots, v_t\}$ is an independent set in $H\}$.

For $t = 1$, we use $\delta(H)$ for $\sigma_1(H)$ and $U_1(H)$. In general, $\sigma_t(H) \geq U_t(H)$. Let

$$\Omega(H) = \{\sigma_t(H), U_t(H)\}.$$

Sufficient conditions involved parameters in $\Omega(H)$ for claw-free graphs to be Hamiltonian have been the subjects of many papers (see [10, 12, 17]). For 2-connected claw-free graph H of order

n, Matthews and Sumner [23] shown that if $\delta(H) \geq (n-2)/3$ H is Hamiltonian; Li [19] shown that if $\delta(H) \geq n/4$, then H is either Hamiltonian or belongs to a family of easily described graphs; Flandrin, et al. [14] shown that if $\sigma_2(H) \geq \frac{2n-5}{3}$ then H is Hamiltonian. For $\sigma_t(H)$ with $t \geq 4$, Favaron, et al. [10] proved the following:

Theorem 1.2. *Let $t \geq 4$ be an integer and let H be a 2-connected claw-free simple graph of order n such that $n \geq 3t^2 - 4t - 7$, $\delta(H) \geq 3t - 4$ and $\sigma_t(H) > n + t^2 - 4t + 7$. Then either H is Hamiltonian or $\theta(cl(H)) \leq t - 1$.*

As a special case of Theorem 1.2, Favaron, et al. [10] shown that a 2-connected claw-free graph H of order $n \geq 77$ with $\delta(H) \geq 14$ and $\sigma_6(H) > n + 19$ is either Hamiltonian or belongs to a well described exception family. With Theorem 1.2 and the help of a computer, Kovářík et al. [17] obtained a result for $\sigma_8(H) > n + 39$ with an exception family that contains 318 infinite classes.

For $\sigma_3(H)$, Liu et al. [22], Zhang [29] and Broersma [3] shown that a 2-connected claw-free graph H of order n with $\sigma_3(H) \geq n - 2$ is Hamiltonian. For condition involved $\sigma_4(H)$ for the hamiltonicity of claw-free graphs, Frydrych proved the following and had a conjecture in [15].

Theorem 1.3 (Frydrych [15])**.** *A 2-connected claw-free simple graph H of order n with $\sigma_4(H) \geq n + 3$ is either Hamiltonian or $cl(H) \in \mathcal{Q}_{2,3}(s_1, s_2, s_3, n)$.*

Conjecture 1.4 (Frydrych [15])**.** *Theorem 1.3 still holds if $\sigma_4(H) \geq n$ and $\delta(H) \geq 3$.*

The condition "$\delta(H) \geq 3$" in Conjecture 1.4 was not in the original statement in [15]. However, it would not be true if $\delta(H) = 2$ as shown by the graph in Fig.1, where $K_s = K_{(n-3)/2}$ and H is a non-hamiltonian claw-free graph of order n with $\delta(H) = 2$, $\sigma_4(H) \geq n + 1$ and $cl(H) \notin \mathcal{Q}_{2,3}(s_1, s_2, s_3, n)$.

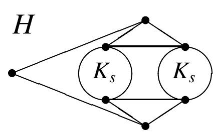

Fig. 1: A nonhamiltonian graph H of order n with $\delta(H) = 2$ and $\sigma_4(H) \geq n + 1$.

For 3-connected claw-free graphs H of order n, Zhang [29] proved that if $\sigma_4(H) \geq n - 3$, then H is Hamiltonian; Wu [27] proved that if $\sigma_3(H) \geq n + 1$, then H is Hamiltonian connected. Settling a conjecture posed in [13], Lai et al. [18] proved the following:

Theorem 1.5 (Lai et al. [18])**.** *A 3-connected claw-free simple graph H of order $n \geq 196$ with $\delta(H) \geq \frac{n+5}{10}$ is either Hamiltonian or $cl(H) \in \mathcal{Q}_P^1(n, \frac{n-15}{10})$.*

By enlarging the exception family, Li [21] improved Theorem 1.5 for such graphs H with $\delta(H) \geq \frac{n+34}{12}$. Solving a conjecture in [21], Chen, et al. in [9] further improved Li's result to $\delta(H) \geq \frac{n+6}{13}$.

For $U_t(H)$ condition on the hamiltonicity of claw-free graphs, the following are known:

Theorem 1.6. *Let H be a k-connected claw-free simple graph of order n. Then each of the following holds:*

(a) *(Bauer, Fan and Veldman [1]) If $k = 2$ and $U_2(H) \geq \frac{2n-5}{3}$, then H is Hamiltonian.*

(b) *(Li and Virlouvet [20]) If $k = 3$ and $U_2(H) \geq \frac{11(n-7)}{21}$, then H is Hamiltonian.*

Theorem 1.6(b) is a special case of the following Theorem.

Theorem 1.7. *(Li and Virlouvet [20]) Let H be a k-connected ($k \geq 3$) claw-free simple graph of order n. If there is some integer t, $t \leq 2k$, such that $U_t(H) \geq \dfrac{t(4k-t+1)}{2k(2k+1)}(n-2k-1)$, then H is Hamiltonian.*

In this paper, we unify and strengthen the results involved $d_t(H) \in \Omega(H)$ above and prove Conjecture 1.4 which is an easy conclusion from the main result.

Let p and t be positive integers and let ϵ be a given number. Let H be a k-connected claw-free graph of order n ($k \geq 2$). For $d_t(H) \in \Omega(H)$, we consider graphs H that satisfy the following:

$$d_t(H) \geq \frac{t(n+\epsilon)}{p}. \tag{1}$$

All the conditions involved $d_t(H) \in \Omega(H)$ in the theorems mentioned above are the special cases of (1) with various given values of p, t, and ϵ.

Let $Q_0(r,k)$ be the family of k-edge-connected K_3-free graphs of order at most r and without an SCT. It is known that $Q_0(5,2) = \{K_{2,3}\}$ and $Q_0(13,3) = \{P\}$ (see Theorem 2.3 in section 2).

For given integer $p > 0$ and a real number ϵ, define

$$N(p,\epsilon) = \max\{36p^2 - 34p - \epsilon(p+1), 20p^2 - 10p - \epsilon(p+1), (3p+1)(-\epsilon - 4p)\}. \tag{2}$$

Our main result is the following:

Theorem 1.8. *Let H be a k-connected claw-free simple graph of order n ($k \geq 2$) and $\delta(H) \geq 3$. For given integers $p \geq t > 0$ and a given number ϵ, if $d_t(H) \geq \dfrac{t(n+\epsilon)}{p}$ where $d_t(H) \in \Omega(H)$ and $n > N(p,\epsilon)$, then either H is Hamiltonian or $cl(H) = L(G)$ where G is an essentially k-edge-connected K_3-free graph without a DCT and G satisfies one of the following:*

(a) *if $k = 2$, G is contractible to a graph in $Q_0(c,2)$ where $c \leq \max\{4p-5, 2p+1\}$;*

(b) *if $k = 3$, G is contractible to a graph in $Q_0(c,3)$ where $c \leq \max\{3p-5, 2p+1\}$.*

It should be known that "G is contractible to a graph in $Q_0(c,k)$" in Theorem 1.8 means that "the reduction G_0' of the core G_0 of G is in $Q_0(c,k)$" which is defined by the Catlin's reduction method given in next section. As applications of Theorem 1.8, we prove the following two theorems.

Theorem 1.9. *Let H be a 2-connected claw-free simple graph of order n with $\delta(H) \geq 3$ and n is sufficiently large. If $d_t(H) \geq \frac{tn}{4}$ where $d_t(H) \in \Omega(H)$ and t is a given integer and $1 \leq t \leq 4$, then either H is Hamiltonian or $cl(H) \in Q_{2,3}(s_1, s_2, s_3, n)$ where $s_1 + s_2 + s_3 + 6 = n$.*

Theorem 1.10. *Let H be a 3-connected claw-free simple graph of order n and n is sufficiently large.*

(a) For a given integer t and $1 \le t \le 10$, if $d_t(H) \ge \frac{t(n+5)}{10}$ where $d_t(H) \in \Omega(H)$ and $\delta(H) \ge 24$, then H is Hamiltonian if and only if $cl(H) \notin Q_P^1(n, \frac{n-15}{10})$.

(b) If $\sigma_{13}(H) \ge n+6$ and $\delta(H) \ge 33$, then H is Hamiltonian if and only if $cl(H) \notin Q_P(n, 1)$.

Remarks. (a) The case for $d_t(H) = \sigma_4(H) \ge n$ of Theorem 1.9 verifies Conjecture 1.4. The case for $d_t(H) = \sigma_3(H) \ge \frac{3n}{4}$ of Theorem 1.9 is an improvement of a "$\sigma_3(H) \ge n-2$" theorem obtained by Liu et al. [22], Zhang [29] and Broersma [3] mentioned above; the case for $d_t(H) = \sigma_2(H) \ge \frac{n}{2}$ is an improvement of a "$\sigma_2(H) \ge \frac{2n-5}{3}$" theorem proved by Flandrin, et al. in [14]; the case $d_t(H) = \sigma_1(H) = \delta(H)$ is a theorem proved by Li in [19]. The case for $d_t(H) = U_t(H)$ with $1 \le t \le 4$ of Theorem 1.9 is an improvement of Theorem 1.6(a).

The case for $d_t(H) = \sigma_t(H)$ of Theorem 1.10(a) is a generalization and improvement of Theorem 1.5. It shows that the conclusion of Theorem 1.5 holds for $\sigma_t(H) \ge \frac{t(n+5)}{10}$ for any $t \in \{1, 2, \cdots, 10\}$. The case for $d_t(H) = \sigma_t(H)$ of Theorem 1.10(b) is an improvement of the results in [18, 21]. The case for $d_t(H) = U_t(H)$ of Theorem 1.10 is an improvement of Theorem 1.6(b) and Theorem 1.7 with $k = 3$.

(b) One can check whether a graph belongs to $Q_{2,3}(s_1, s_2, s_3, n) \cup Q_P^1(n, \frac{n-15}{10})$ in polynomial time. For graphs H satisfying Theorems 1.9 or 1.10(a), it can be determined in polynomial time if H is Hamiltonian. For Theorem 1.10(b), a graph given in [9] shows that the result is best possible in the sense that $p = 13$ cannot be replaced by $p = 14$.

(c) For given p, t, ϵ and k, comparing to the family of k-connected claw-free graphs of order n with $d_t(H) \ge \frac{t(n+\epsilon)}{p}$ where $d_t(H) \in \Omega(H)$, the number of graphs in $Q_0(4p-5, 2) \cup Q_0(3p-5, 3)$ is fixed and can be determined in a constant time (independent on n). In some sense, Theorem 1.8 shows that only a finite number of k-connected claw-free graphs H with $d_t(H) \ge \frac{t(n+\epsilon)}{p}$ are non-Hamiltonian. One may obtain new improvements to Theorems 1.10 and 1.9 by enlarging the number of exceptions with the help of a computer.

(d) Faudree et al. [11] define the *generalized t-degree*, $\delta_t(H)$, of a graph H by

$$\delta_t(H) = \min\{|\bigcup_{i=1}^t N_H(x_i)| \mid \{x_1, x_2, \cdots, x_t\} \text{ is a } t\text{-subset in } H\}$$

Since $\sigma_t(H) \ge U_t(H) \ge \delta_t(H)$, Theorems 1.8, 1.9 and 1.10 are also true for $d_t(H) = \delta_t(H)$.

The rest of this paper is organized as follows. In Section 2, we give a brief discussion of Ryjáček closure concept and Catlin's reduction method. In Section 3, we prove a technical lemma which will be needed in our proofs. The proof of Theorem 1.8 is given in section 4. In Section 5, we prove a lemma on the properties of reduced graph related to σ_t condition. The proofs of Theorems 1.9 and 1.10 are given in the last section.

2 Ryjáček closure concept and Catlin's reduction Method

The following is a main theorem of Ryjáček closure concept.

Theorem 2.1. *(Ryjáček [25]). Let H be a claw-free graph and $cl(H)$ its closure. Then*

(a) $cl(H)$ is well defined, and $\kappa(cl(H)) \geq \kappa(H)$;

(b) there is a K_3-free graph G such that $cl(H) = L(G)$;

(c) both graphs H and $cl(H)$ have the same circumference.

It is known that a connected line graph $H \neq K_3$ has a unique graph G with $H = L(G)$. We call G the preimage graph of H. For a claw-free graph H, the closure $cl(H)$ of H can be obtained in polynomial time [25] and the preimage graph of a line graph can be obtained in linear time [24]. We can compute G efficiently for $cl(H) = L(G)$. Thus, with Theorems 1.1 and 2.1, finding a Hamiltonian cycle in a claw-free graph H is equivalent to finding a DCT in the preimage graph G of $cl(H)$.

Next, we give a brief discussion on Catlin's reduction method.

Let G be a connected multigraph. For $X \subseteq E(G)$, the *contraction* G/X is the graph obtained from G by identifying the two ends of each edge $e \in X$ and deleting the resulting loops. G/X may not be simple. If Γ is a connected subgraph of G, then Γ is contracted to a vertex in G/Γ and we write G/Γ for $G/E(\Gamma)$.

Let $O(G)$ be the set of vertices of odd degree in G. A graph G is *collapsible* if for every even subset $R \subseteq V(G)$, there is a spanning connected subgraph Γ_R of G with $O(\Gamma_R) = R$. K_1 is regarded as a collapsible and supereulerian graph. We use $\mathcal{CL}$ to denote the family of collapsible graphs.

In [4], Catlin showed that every graph G has a unique collection of maximal collapsible subgraphs $\Gamma_1, \Gamma_2, \cdots, \Gamma_c$. The *reduction* of G is $G' = G/(\cup_{i=1}^{c} \Gamma_i)$, the graph obtained from G by contracting each Γ_i into a single vertex v_i ($1 \leq i \leq c$). For a vertex $v \in V(G')$, there is a unique maximal collapsible subgraph $\Gamma_0(v)$ such that v is the contraction image of $\Gamma_0(v)$ and $\Gamma_0(v)$ is the *preimage* of v. A vertex $v \in V(G')$ is *contracted vertex* if $\Gamma_0(v) \neq K_1$. A graph G is *reduced* if $G' = G$.

Theorem 2.2. *(Catlin, et al. [4, 5]). Let G be a connected graph and let G' be the reduction of G.*

(a) $G \in \mathcal{CL}$ if and only if $G' = K_1$, and $G \in \mathcal{SL}$ if and only if $G' \in \mathcal{SL}$.

(b) G has a DCT if and only if G' has a DCT containing all the contracted vertices of G'.

(c) If G is a reduced graph, then G is simple and K_3-free with $\delta(G) \leq 3$. For any subgraph Ψ of G, Ψ is reduced and either $\Psi \in \{K_1, K_2, K_{2,t} (t \geq 2)\}$ or $|E(\Psi)| \leq 2|V(\Psi)| - 5$.

Let P_{14} be the graph obtained from P by replacing a vertex v in P by a $K_{2,3}$ in the way that the three edges incident with v in P are incident with the three degree 2 vertices in $K_{2,3}$, respectively.

Some facts on reduced graphs are summarized in the following theorem.

Theorem 2.3. *Let G be a connected reduced graph of order n. Then each of the following holds:*
(a) If $G \notin \mathcal{SL}$ and $\kappa'(G) \geq 2$, then $n \geq 5$ and $n = 5$ only if $G = K_{2,3}$.
(b) ([7]) For $1 < n \leq 9$, if $\kappa'(G) \geq 2$, then $|D_2(G)| \geq 3$.
(c) ([7]) If $\kappa'(G) \geq 3$ and $n \leq 14$, then either $G \in \mathcal{SL}$ or $G \in \{P, P_{14}\}$.
(d) ([7]) If $\kappa'(G) \geq 3$ and $n = 15$, then either $G \in \mathcal{SL}$ or G is 2-connected, 3-edge-connected and essentially 4-edge-connected graph with girth at least 5 and $V(G) = D_3(G) \cup D_4(G)$ where $|D_4(G)| = 3$ and $D_4(G)$ is an independent set.
(e) ([6]) Let G be a connected reduced graph of order n with $\delta(G) \geq 2$. Let M be a maximum matching in G and $|D_2(G)| = l$, and $G \neq K_{2,a}$ ($a \geq 2$). Then $|M| \geq \min\{\frac{n-1}{2}, \frac{n+5-l}{3}\}$.

Let H be a k-connected claw-free graph with $\delta(H) \geq 3$ ($k \in \{2, 3\}$). By Theorem 2.1, there is a K_3-free graph G such that $cl(H) = L(G)$. By the definition of $cl(H)$, $V(cl(H)) = V(H)$ and $d_{cl(H)}(v) \geq d_H(v)$ for any $v \in V(cl(H))$ and so $\delta(cl(H)) \geq \delta(H) \geq 3$. For an edge $e = xy$ in G, let v_e be the vertex in $cl(H)$ defined by e in G. Then $d_{cl(H)}(v_e) + 2 = d_G(x) + d_G(y)$. Thus, if $cl(H) = L(G)$ is k-connected graph with $\delta(cl(H)) \geq 3$, then G is essentially k-edge-connected with $\overline{\sigma}_2(G) \geq 5$.

Let G be an essentially k-edge-connected graph with $\overline{\sigma}_2(G) \geq 5$, where $k \in \{2, 3\}$. Then $D_1(G) \cup D_2(G)$ is an independent set. Let E_1 be the set of pendant edges in G. For each $x \in D_2(G)$, there are two edges e_x^1 and e_x^2 incident with x. Let $X_2(G) = \{e_x^1 \, | x \in D_2(G)\}$. Define

$$G_0 = G/(E_1 \cup X_2(G)) = (G - D_1(G))/X_2(G).$$

In other words, G_0 is obtained from G by deleting the vertices in $D_1(G)$ and replacing each path of length 2 whose internal vertex is a vertex in $D_2(G)$ by an edge.

Let $X = D_1(G) \cup D_2(G)$. In [28], G_0 is denoted by $I_X(G)$. In [26], Shao defined G_0 for essentially 3-edge-connected graphs G. Following [26], we call G_0 the *core* of G. Note that even G is simple, G_0 may not be simple.

The vertex set $V(G_0)$ is regarded as a subset of $V(G)$. A vertex in G_0 is nontrivial if it is obtained by contracting some edges in $E_1 \cup X_2(G)$ or it is adjacent to a vertex in $D_2(G)$ in G. For instance, if $x \in D_2(G)$ and $N_G(x) = \{u, v\}$ and if u_x in G_0 is obtained by contracting the edge ux, then both u_x and v are nontrivial in G_0 although u_x is a contracted vertex and v is not a contracted vertex in G_0. When we say u_x is adjacent to a vertex in $D_2(G)$, we regard u_x as vertex u in this case. Since $\overline{\sigma}_2(G) \geq 5$, all vertices in $D_2(G_0)$ are nontrivial

Let G_0' be the reduction of G_0. For a vertex $v \in V(G_0')$, let $\Gamma_0(v)$ be the maximum collapsible preimage of v in G_0 and let $\Gamma(v)$ be the preimage of v in G which is the graph induced by edges in $E(\Gamma_0(v))$ and some edges in $E_1 \cup X_2(G)$. A vertex v in G_0' is a *nontrivial vertex* if v is a contracted vertex (i.e., $|E(\Gamma(v))| \geq 1$) or v is adjacent to a vertex in $D_2(G)$.

For a vertex x in $V(\Gamma(v))$, let $I(x)$ be the set of edges in $E(G_0')$ that are incident with x in G. Let $i(x) = |I(x)|$. Then $i(x)$ is the number of edges in $E(G_0')$ that are incident with x in G. For any

$x \in V(\Gamma(v))$,

$$i(x) \le \sum_{x \in V(\Gamma(v))} i(x) = d_{G'_0}(v), \text{ and } \mathrm{d_G(x) \le i(x) + |V(\Gamma(v))| - 1 \le i(x) + |E(\Gamma(v))|}. \tag{3}$$

Using Theorem 2.2, Veldman [28] and Shao [26] proved the following:

Theorem 2.4. *Let G be a connected and essentially k-edge-connected graph ($k \ge 2$) with $\overline{\sigma}_2(G) \ge 5$ and $L(G)$ is not complete. Let G_0 be the core of graph G. Let G'_0 be the reduction of G_0. Then each of the following holds:*
(a) G_0 is well defined, nontrivial and $\kappa'(G'_0) \ge \kappa'(G_0) \ge \min\{3, k\}$.
(b) (Lemma 5 [28]) G has a DCT if and only if G'_0 has a DCT containing all the nontrivial vertices.

In the rest of the paper, we will use the following notation related to G'_0:

- $S_0 = \{v \in V(G'_0) \mid v \text{ is a nontrivial vertex in } G'_0\}$;
- $S_1 = \{v \in S_0 \mid |E(\Gamma(v))| \ge 1\}$;
- $S_2 = S_0 - S_1$, the set of vertices v with $\Gamma(v) = K_1$ and adjacent to some vertices in $D_2(G)$;
- $V_0 = V(G'_0) - S_1$, the set of vertices v with $\Gamma(v) = K_1$ in G which includes S_2;
- $\Phi_0 = G'_0[V_0]$;
- M_0 is a maximum matching in Φ_0, and V_{M_0} is the vertex set of M_0;
- $U_0 = V_0 - V_{M_0}$ and so $V(G'_0) = S_1 \cup V_{M_0} \cup U_0$.

Since $\overline{\sigma}_2(G) \ge 5$, by the definition of G'_0, $D_2(G'_0) \subseteq S_1$.

3 A Technical Lemma

Since $\sigma_t(H) \ge U_t(H)$, $U_t(H) \ge \dfrac{t(n+\epsilon)}{p}$ implies $\sigma_t(H) \ge \dfrac{t(n+\epsilon)}{p}$. It will be sufficient to prove Theorems 1.8, 1.9 and 1.10 for σ_t. We prove the following lemma for σ_t only.

Lemma 3.1. *Let H be the graph satisfying Theorem 1.8 with $cl(H) = L(G)$. Let G_0 and G'_0 be the graphs related to G defined in section 2. For each $v \in V(G'_0)$, let $\Gamma(v)$ be the preimage of v in G. Then each of the following holds:*

(a) Let M be a matching in G with $|M| \ge t$. Then

$$|M|\frac{\sigma_t(H) + 2t}{t} \le \sum_{xy \in M} (d_G(x) + d_G(y)). \tag{4}$$

(b) *Let $V_r \subseteq S_1$ be a r-subset of S_1 in G_0'. Let M_b' be a matching of size b in G_0'. Let $V(M_b')$ be the vertex set of M_b'. Suppose that $V_r \cap V(M_b') = \emptyset$. If $|V_r| + |M_b'| = r + b \geq t$, then*

$$\sum_{v \in V_r}(|V(\Gamma(v))| + d_{G_0'}(v)) + \sum_{xy \in M_b'}(|V(\Gamma(x))| + |V(\Gamma(y))| + d_{G_0'}(x) + d_{G_0'}(y)) \geq \frac{(r+b)(\sigma_t(H)+2t)}{t} + 2b.$$

(c) *If H satisfies (1), then $|D_2(G_0')| \leq p$ when $n > -\epsilon(p+1)$.*

Proof. (a) Let $m = |M|$ and let M_t be a t-subset of M such that for any $ab \in M - M_t$,

$$\max_{xy \in M_t}\{d_G(x) + d_G(y)\} \leq d_G(a) + d_G(b). \tag{5}$$

Let A_t be the t-vertex set in $V(cl(H)) = V(H)$ defined by the edges in M_t. Then A_t is a t-independent set in $cl(H)$ (as well as in H). Since $d_H(v_e) \leq d_{cl(H)}(v_e)$,

$$\sigma_t(H) + 2t \leq \sum_{v_e \in A_t}(d_H(v_e) + 2) \leq \sum_{v_e \in A_t}(d_{cl(H)}(v_e) + 2) = \sum_{e = xy \in M_t}(d_G(x) + d_G(y)). \tag{6}$$

For $ab \in M - M_t$, by (6) and (5),

$$\frac{\sigma_t(H) + 2t}{t} \leq \frac{\sum_{x_i y_i \in M_t}(d_G(x_i) + d_G(y_i))}{t} \leq \frac{t(d_G(a) + d_G(b))}{t} = d_G(a) + d_G(b). \tag{7}$$

By (6), (7) and $m = |M|$,

$$\begin{aligned}\sum_{xy \in M}(d_G(x) + d_G(y)) &= \sum_{x_i y_i \in M_t}(d_G(x_i) + d_G(y_i)) + \sum_{ab \in M - M_t}(d_G(a) + d_G(b)) \\ &\geq \sigma_t(H) + 2t + (m-t)(\frac{\sigma_t(H)+2t}{t}) = m\frac{\sigma_t(H)+2t}{t}.\end{aligned}$$

Case (a) is proved.

(b) Let $V_r = \{v_1, v_2, \cdots, v_r\}$ and let $\Gamma(v_i)$ be the preimage of v_i ($1 \leq i \leq r$) in G. Since $V_r \subseteq S_1$, $\Gamma(v_i)$ is nontrivial. Let $x_i y_i$ be an edge in $\Gamma(v_i)$. Let $M_r = \{x_i y_i \mid 1 \leq i \leq r\}$. For each $x_i y_i \in M_r$, since G is K_3-free, $N_G(x_i) \cap N_G(y_i) = \emptyset$ and $N_G(x_i) \cup N_G(y_i) \subseteq I(x_i) \cup I(y_i) \cup V(\Gamma(v_i))$. By (3),

$$d_G(x_i) + d_G(y_i) \leq i(x_i) + i(y_i) + |V(\Gamma(v_i))| \leq d_{G_0'}(v_i) + |V(\Gamma(v_i))|. \tag{8}$$

For each $e = xy \in M_b'$, let $\Gamma(x)$ and $\Gamma(y)$ be the preimages of x and y in G, respectively. Then there is a vertex u in $V(\Gamma(x))$ and a vertex v in $V(\Gamma(y))$ such that $uv = e$, the edge in G corresponding to xy in G_0'. Let $M_b^0 = \{uv \mid u \in V(\Gamma(x)), v \in V(\Gamma(y))$ for each $xy \in M_b'\}$. M_b^0 is a b-matching in G.

For $uv \in M_b^0$ with $u \in V(\Gamma(x))$ and $v \in V(\Gamma(y))$,

$$d_G(u) \leq d_{G_0'}(x) + |V(\Gamma(x))| - 1 \text{ and } d_G(v) \leq d_{G_0'}(y) + |V(\Gamma(y))| - 1. \tag{9}$$

For each $uv \in M_b^0$ and its corresponding edge $xy \in M_b'$, by (9)

$$d_G(u) + d_G(v) \leq d_{G_0'}(x) + d_{G_0'}(y) + |V(\Gamma(x))| + |V(\Gamma(y))| - 2. \tag{10}$$

Since $V_r \cap V(M'_b) = \emptyset$, $M = M_r \cup M_b^0$ is a matching in G with $m = |M| = r + b \geq t$. By (4),

$$\sum_{xy \in M} (d_G(x) + d_G(y)) \geq |M| \frac{\sigma_t(H) + 2t}{t}. \tag{11}$$

Since $M = M_r \cup M_b^0$ and $b = |M'_b|$, by (11), (8) and (10)

$$\begin{aligned}
|M| \frac{\sigma_t(H) + 2t}{t} &\leq \sum_{xy \in M} d_G(x) + d_G(y) = \sum_{x_i y_i \in M_r} (d_G(x_i) + d_G(y_i)) + \sum_{uv \in M_b^0} (d_G(u) + d_G(v)) \\
&\leq \sum_{v_i \in V_r} (d_{G'_0}(v_i) + |V(\Gamma(v_i))|) + \sum_{xy \in M'_b} (d_{G'_0}(x) + d_{G'_0}(y) + |V(\Gamma(x))| + |V(\Gamma(y))| - 2); \\
|M| \frac{\sigma_t(H) + 2t}{t} + 2b &\leq \sum_{v_i \in V_r} (d_{G'_0}(v_i) + |V(\Gamma(v_i))|) + \sum_{xy \in M'_b} (d_{G'_0}(x) + d_{G'_0}(y) + |V(\Gamma(x))| + |V(\Gamma(y))|).
\end{aligned}$$

Case (b) is proved.

(c). By way of contradiction, suppose that $r = |D_2(G'_0)| > p$. Since $\overline{\sigma}_2(G) \geq 5$, $D_2(G'_0) \subseteq S_1$. Let $V_r = D_2(G'_0)$. By $p \geq t$ and (b) above with $M'_b = \emptyset$ and $d_{G'_0}(v_i) = 2$ for $v_i \in D_2(G'_0)$,

$$\begin{aligned}
\sum_{v_i \in V_r} |V(\Gamma(v_i))| + 2r &= \sum_{v_i \in V_r} (|V(\Gamma(v_i))| + d_{G'_0}(v_i)) \geq \frac{r(\sigma_t(H) + 2t)}{t}; \\
\sum_{v_i \in V_r} |V(\Gamma(v_i))| &\geq \frac{r\sigma_t(H)}{t}.
\end{aligned} \tag{12}$$

Since G is not a tree, $|E(G)| \geq |V(G)|$. Since $|V(G)| \geq \sum_{v \in V_r} |V(\Gamma(v))|$, by (12), (1) and $n = |E(G)|$

$$\begin{aligned}
n = |E(G)| &\geq \sum_{v \in V_r} |V(\Gamma(v))| \geq r \frac{\sigma_t(H)}{t} \geq r \frac{\frac{t(n+\epsilon)}{p}}{t} = \frac{r}{p}(n + \epsilon); \\
r &\leq p + \frac{-\epsilon p}{n + \epsilon}.
\end{aligned}$$

Thus, when $n > -\epsilon(p + 1)$, $|D_2(G'_0)| = r \leq p$. Case (c) is proved. □

4 Proof of Theorem 1.8

Proof of Theorem 1.8. Suppose that H is not Hamiltonian. By Theorem 2.1, there is an essentially k-edge-connected K_3-free graph G such that the closure $cl(H) = L(G)$. Then $L(G)$ is not completed and $|E(G)| = n = |V(H)|$. Let G_0 be the core of G. Let G'_0 be the reduction of G_0 and $c = |V(G'_0)|$. By Theorem 2.4, G'_0 does not have an SCT and $\kappa'(G'_0) \geq \kappa'(G_0) \geq \min\{3, k\}$. For $k = 2$, let $r = |D_2(G'_0)|$.

If $G'_0 = K_{2,a}$, then by Lemma 3.1(c), $a = |D_2(G'_0)| \leq p$. Theorem 1.8(a) holds for this case.

Next, we assume $G'_0 \neq K_{2,a}$. Let M be a maximum matching in G'_0. By Theorem 2.3(e)

$$c \leq \max\{3|M| + r - 5, 2|M| + 1\}. \tag{13}$$

Case 1. $|M| \le t-1$. By (13), $c \le \max\{3t+r-8, 2t-1\}$. Since $t \le p$, if $k=3$, $c \le \max\{3p-8, 2p-1\}$; if $k=2$, by Lemma 3.1(c), $r = |D_2(G_0')| \le p$, $c \le \max\{4p-8, 2p-1\}$. Theorem 1.8(a) holds.

Case 2. $|M| \ge t$. Let $m = |M|$. Note that an edge $e = xy$ in M can be viewed as an edge $e = uv$ in G and

$$d_G(u) + d_G(v) \le |V(\Gamma(x))| + |V(\Gamma(y))| + d_{G_0'}(x) + d_{G_0'}(y) - 2. \tag{14}$$

Let $M_G = \{uv \mid uv$ is an edge in G corresponding to an edge xy in $M\}$. Then M_G is a matching with $|M_G| = |M| \ge t$. By Lemma 3.1(a) and (14),

$$\frac{m(\sigma_t(H)+2t)}{t} + 2m \le \sum_{uv\in M_G} (d_G(u) + d_G(v) + 2);$$

$$\frac{m(\sigma_t(H)+4t)}{t} \le \sum_{xy\in M} (|V(\Gamma(x))| + |V(\Gamma(y))| + d_{G_0'}(x) + d_{G_0'}(y)) \le |V(G)| + \sum_{v\in V(G_0')} d_{G_0'}(v). \tag{15}$$

Since G is not a tree, $|E(G)| \ge |V(G)|$. By (1), (15) and by $2|E(G_0')| = \sum_{v\in V(G_0')} d_{G_0'}(v)$,

$$m(\frac{n+\epsilon}{p} + 4) \le \frac{m(\sigma_t(H)+4t)}{t} \le |V(G)| + \sum_{v\in V(G_0')} d_{G_0'}(v) \le |E(G)| + 2|E(G_0')|. \tag{16}$$

Claim 1. $|E(G_0')| \le \max\{20p-15, 12p-3\}$.

By (1), (16), and by $|E(G_0')| \le |E(G)| = n$, $m(\frac{n+\epsilon}{p} + 4) \le |E(G)| + 2|E(G_0')| \le 3n$, and so

$$m \le 3p - \frac{3p(\epsilon+4p)}{n+\epsilon+4p}.$$

Therefore, $m \le 3p$ since $n > N(p,\epsilon) \ge (3p+1)(-\epsilon-4p)$. By (13) and $r \le p$, $c \le \max\{3m+r-5, 2m+1\} \le \max\{9p+r-5, 6p+1\} \le \max\{10p-5, 6p+1\}$. By Theorem 2.2 and $G_0' \ne K_{2,a}$,

$$|E(G_0')| \le 2|V(G_0')| - 5 \le 2\max\{10p-5, 6p+1\} - 5 = \max\{20p-15, 12p-3\}. \tag{17}$$

Claim 1 is proved.

By (16), (17), and by $|V(G)| \le |E(G)| = n$,

$$\begin{aligned}
m(\frac{n+\epsilon}{p} + 4) &\le |E(G)| + 2|E(G_0')| \le n + 2\max\{20p-15, 12p-3\};\\
m &\le \frac{np + 2p\max\{20p-15, 12p-3\}}{n+\epsilon+4p} = p + \frac{p\max\{40p-30, 24p-6\} - (\epsilon+4p)p}{n+\epsilon+4p}\\
&\le p + \frac{p\max\{36p-30-\epsilon, 20p-6-\epsilon\}}{n+\epsilon+4p}.
\end{aligned}$$

Thus, $m \le p$ since $n > N(p,\epsilon) \ge p\max\{36p-30-\epsilon, 20p-6-\epsilon\} - \epsilon - 4p$. By (13) and $r \le p$, if $k=2$, $c \le \max\{4p-5, 2p+1\}$; if $k=3$, $c \le \max\{3p-5, 2p+1\}$. Theorem 1.8 is proved. □

Remark. The expression $N(p,\epsilon)$ defined by (2) is for the convenience in the proofs above. To avoid a lengthy case by case checking, we did not make efforts to get a best possible bound for this quantity.

5 Properties of G_0' for graphs G satisfying Theorem 1.8

The following lemma will be needed for the proofs of Theorems 1.9 and 1.10

Lemma 5.1. *Let H be a graph of order n that satisfies Theorem 1.8 with the given numbers k, p, t and ϵ, where $k \in \{2,3\}$, $p \ge 3(k-1)$ and $p \ge t$. Suppose that H is nonhamiltonian with $cl(H) = L(G)$. Let G_0 be the core of G. Let G_0' the reduction of G_0. Let S_0, S_1, S_2, M_0, V_0 and U_0 be the sets defined in Section 2. If $n > N(p,\epsilon)$ and $G_0' \ne K_{2,a}$, then each of the following holds:*

(a) $|S_1| + |M_0| \le p$.

(b) *If* $|S_1| + |M_0| = p$, *then* $|E(G_0')| \ge 2p + \epsilon - |S_1| + \sum_{v \in U_0} d_G(v)$. *Furthermore, if* $|M_0| = 0$, *then* $V(G_0') = S_1 \cup U_0$, $|E(G_0')| \ge \epsilon + p + \sum_{v\in U_0} d_G(v)$ *and* $|V(G_0')| \le 2p - \epsilon - 5$.

(c) $|U_0| \le 2|S_1| + 3|M_0| - 5$ *and* $|V(G_0')| \le 3|S_1| + 5|M_0| - 5$.

(d) *If* $\delta(H) \ge 3p - 6$ *when* $k = 3$ *or if* $\delta(H) \ge 4p - 6$ *when* $k = 2$, *then* $M_0 = \emptyset$ *and* $S_2 = \emptyset$.

Proof. Since H is nonhamiltonian, by Theorem 2.4, G_0' does not have a DCT containing S_0. Since $p \ge (k-1)3$, $\max\{4p-5, 2p+1\} = 4p-5$ when $k = 2$ and $\max\{3p-5, 2p+1\} = 3p-5$ when $k = 3$. By Theorem 2.2 and $G_0' \ne K_{2,a}$, and by Theorem 1.8,

$$|E(G_0)| \le 2|V(G_0')| - 5 \le \left\{ \begin{array}{ll} 6p-15 & \text{if } k=3; \\ 8p-15 & \text{if } k=2, \end{array} \right\} \le 8p-15. \tag{18}$$

(a) Let $s = |S_1|$ and $m = |M_0|$. If $s + m < t$, then we are done. Thus, we assume $s + m \ge t$.

Since $S_1 \cap V_{M_0} = \emptyset$, by Lemma 3.1(b) with $|S_1| + |M_0| = s + m \ge t$,

$$(s+m)\frac{\sigma_t(H)+2t}{t} + 2m \le \sum_{v_i \in S_1} (d_{G_0'}(v_i) + |V(\Gamma(v_i))|) + \sum_{xy\in M_0} (d_{G_0'}(x) + d_{G_0'}(y) + |V(\Gamma(x))| + |V(\Gamma(y))|)$$

$$(s+m)\frac{\sigma_t(H)+2t}{t} + 2m \quad \le \quad \sum_{v_i\in S_1\cup V_{M_0}} d_{G_0'}(v) + \sum_{v_i\in S_1} |V(\Gamma(v_i))| + \sum_{xy\in M_0} (|V(\Gamma(x))| + |V(\Gamma(y))|). \tag{19}$$

For each $xy \in M_0$, since x and y are vertices in V_0, $|V(\Gamma(x))| = |V(\Gamma(y))| = 1$. By (19),

$$(s+m)\frac{\sigma_t(H)+2t}{t} - \sum_{v_i\in S_1\cup V_{M_0}} d_{G_0'}(v) \le \sum_{v_i\in S_1} |V(\Gamma(v_i))|. \tag{20}$$

Since $|E(\Gamma(v))| \ge |V(\Gamma(v))| - 1$ for $v \in S_1$, by (20), $s = |S_1|$ and $n = |E(G)|$, we have

$$\begin{aligned} |E(G)| &= \sum_{v\in S_1} |E(\Gamma(v))| + |E(G_0')| \ge \sum_{v\in S_1} (V(\Gamma(v))| - 1) + |E(G_0')| \\ &\ge \sum_{v\in S_1} |V(\Gamma(v))| - |S_1| + |E(G_0')|; \\ n &\ge \left((s+m)\frac{\sigma_t(H)+2t}{t} - \sum_{v_i\in S_1\cup V_{M_0}} d_{G_0'}(v) \right) - s + |E(G_0')|. \end{aligned} \tag{21}$$

Since $V(G_0') = S_1 \cup V_{M_0} \cup U_0$, $2|E(G_0')| = \sum_{v\in S_1\cup V_{M_0}} d_{G_0'}(v) + \sum_{v\in U_0} d_{G_0'}(v)$.

$$\sum_{v\in S_1\cup V_{M_0}} d_{G_0'}(v) = 2|E(G_0')| - \sum_{v\in U_0} d_{G_0'}(v). \tag{22}$$

By (21), (22) and (1),

$$\begin{aligned} n &\geq \left((s+m)\frac{\sigma_t(H)+2t}{t} - \left(2|E(G_0')| - \sum_{v\in U_0} d_{G_0'}(v)\right)\right) - s + |E(G_0')|; \\ n &\geq (s+m)(\frac{n+\epsilon}{p}+2) - |E(G_0')| + \sum_{v\in U_0} d_{G_0'}(v) - s; \\ n + |E(G_0')| + s &\geq (s+m)(\frac{n+\epsilon}{p}+2) + \sum_{v\in U_0} d_{G_0'}(v) \geq (s+m)(\frac{n+\epsilon}{p}+2). \end{aligned} \tag{23}$$

By (23) and by (18) and $s \leq |V(G_0')| \leq 4p-5$,

$$s+m \leq \frac{p(n+|E(G_0')|+s)}{n+\epsilon+2p} \leq \frac{p(n+12p-20)}{n+\epsilon+2p} = p + \frac{p(10p-20-\epsilon)}{n+\epsilon+2p}.$$

Thus, $(s+m) \leq p$ since $n > N(p,\epsilon) > 10p^2 - 22p - (p+1)\epsilon$. Case (a) is proved.

(b) Since $s+m = p$, by (23),

$$\begin{aligned} n + |E(G_0')| + s &\geq (s+m)(\frac{n+\epsilon}{p}+2) + \sum_{v\in U_0} d_{G_0'}(v) = p(\frac{n+\epsilon}{p}+2) + \sum_{v\in U_0} d_{G_0'}(v); \\ n + |E(G_0')| + s &\geq n + \epsilon + 2p + \sum_{v\in U_0} d_{G_0'}(v); \\ |E(G_0')| &\geq \epsilon + 2p - s + \sum_{v\in U_0} d_{G_0'}(v). \end{aligned} \tag{24}$$

The first part of case (b) is proved.

If $|M_0| = 0$, then $V_{M_0} = \emptyset$ and $|S_1| = p$. Since $D_2(G_0') \subseteq S_1$, $d_{G_0'}(v) \geq 3$ for any $v \in U_0$. By (24),

$$|E(G_0')| \geq \epsilon + p + \sum_{v\in U_0} d_{G_0'}(v) \geq \epsilon + p + 3|U_0|. \tag{25}$$

Since $G_0' \neq K_{2,a}$, by Theorem 2.2, $|E(G_0')| \leq 2|V(G_0')| - 5 = 2(|S_1| + |U_0|) - 5$. By (25) and $|S_1| = p$,

$$\begin{aligned} \epsilon + p + 3|U_0| &\leq |E(G_0')| \leq 2(|S_1| + |U_0|) - 5 = 2p + 2|U_0| - 5; \\ |U_0| &\leq p - 5 - \epsilon. \end{aligned}$$

Thus, $|V(G_0')| = p + |U_0| \leq 2p - 5 - \epsilon$. Case (b) is proved.

(c) Let Φ_1 be the subgraph in G_0' induced by the edges in M_0 and the edges between U_0 and $S_1 \cup V_{M_0}$. Then $V(\Phi_1) = V(G_0')$ and $|E(\Phi_1)| \leq |E(G_0')|$. Since $D_2(G_0') \subseteq S_1$, $d_{G_0'}(v) \geq 3$ for $v \in U_0$. Then

$|E(\Phi_1)| \geq 3|U_0| + |M_0|$. Since $G_0' \neq K_{2,a}$, by Theorem 2.2, $|E(G_0')| \leq 2|V(G_0')| - 5$. Since $|E(\Phi_1)| \leq |E(G_0')|$ and $|V_{M_0}| = 2|M_0|$,

$$\begin{aligned} 3|U_0| + |M_0| &\leq |E(\Phi_1)| \leq 2|V(G_0')| - 5 = 2(|S_1| + |V_{M_0}| + |U_0|) - 5 = 2|S_1| + 4|M_0| + 2|U_0| - 5; \\ |U_0| &\leq 2|S_1| + 3|M_0| - 5. \end{aligned}$$

Therefore, $|V(G_0')| = |S_1| + |V_{M_0}| + |U_0| \leq 3|S_1| + 5|M_0| - 5$.

(d) If $M_0 \neq \emptyset$, let xy be an edge in M_0. Then $\Gamma(x) = \Gamma(y) = K_1$ in G. Thus, $d_G(x) + d_G(y) = d_{G_0'}(x) + d_{G_0'}(y)$. Since G_0' is K_3-free, $N_{G_0'}(x) \cup N_{G_0'}(y) \subseteq V(G_0')$ and $N_{G_0'}(x) \cap N_{G_0'}(y) = \emptyset$. $d_{G_0'}(x) + d_{G_0'}(y) \leq |V(G_0')|$. Hence, $\delta(H) + 2 = \overline{\sigma}_2(G_0') \leq d_G(x) + d_G(y) = d_{G_0'}(x) + d_{G_0'}(y) \leq |V(G_0')|$.

If $S_2 \neq \emptyset$, let $u \in S_2$. Then u is adjacent to a vertex $v \in D_2(G)$ and $\Gamma(u) = K_1$. Since G_0' is 2-edge-connected and K_3-free, $d_G(u) = d_{G_0'}(u) \leq |V(G_0')| - 2$. $\delta(H) + 2 = \overline{\sigma}_2(G_0') \leq d_G(u) + d_G(v) = d_{G_0'}(u) + 2 \leq |V(G_0')| - 2 + 2 = |V(G_0')|$. Thus, if $M_0 \neq \emptyset$ or $S_2 \neq \emptyset$,

$$\delta(H) \leq |V(G_0')| - 2. \tag{26}$$

By Theorem 1.8. $|V(G_0')| \leq 3p - 5$ if $k = 3$ and $|V(G_0')| \leq 4p - 5$ if $k = 2$. By (26)

$$\delta(H) \leq |V(G_0')| - 2 \leq \begin{cases} 3p - 7 & \text{if } k = 3; \\ 4p - 7 & \text{if } k = 2, \end{cases}$$

a contradiction. Thus, $M_0 = \emptyset$ and $S_2 = \emptyset$. Case (d) is proved. □

6 Proofs of Theorem 1.9 and Theorem 1.10

Proof of Theorem 1.9. This is the special case of Theorem 1.8 with $p = 4$, $1 \leq t \leq 4$ and $\epsilon = 0$. Suppose that H is not Hamiltonian. By Theorem 2.1, $cl(H) = L(G)$ where G is an essentially 2-edge-connected K_3-free graph with $|E(G)| = n$. By Theorem 1.1, G does not have a DCT. Let G_0' be the reduction of G_0. Since $\kappa'(G_0') \geq 2$, by Theorems 2.2(c) and 1.8, $|E(G_0')| \leq 2|V(G_0')| - 4 \leq 2(4p - 5) - 4 = 18$. Note that $G_0' \notin \mathcal{SL}$, by Theorem 2.3(a) $|V(G_0')| \geq 5$.

Let S_0, S_1, M_0 and U_0 be the sets defined above. By Theorem 2.4, G_0' does not have a DCT containing S_0. When $n > 18$, $|E(G_0')| < |E(G)|$. Thus, $|S_1| \geq 1$. By Lemma 5.1, $|S_1| + |M_0| \leq 4$.

Case 1. $G_0' \neq K_{2,a}$.

If $|S_1| + |M_0| \leq 3$, then $|M_0| \leq 2$. By Lemma 5.1, $|V(G_0')| \leq 3|S_1| + 5|M_0| - 5 = 4 + 2|M_0| \leq 8$. By Theorem 2.3(b), $|D_2(G_0')| \geq 3$. Then $|S_1| \geq |D_2(G_0')| \geq 3$. Therefore, $|M_0| = 0$. It follows that $|V(G_0')| \leq 3|S_1| + 5|M_0| - 5 = 4$, contrary to that $|V(G_0')| \geq 5$.

Thus, $|S_1|+|M_0| = 4$. By Lemma 5.1(b) with $p = 4$ and $\epsilon = 0$, and by $|U_0| = |V(G_0')|-|S_1|-2|M_0|$,

$$|E(G_0')| \geq 8 - |S_1| + 3|U_0| \geq 3|V(G_0')| + 8 - 4|S_1| - 6|M_0|. \tag{27}$$

By Theorem 2.2 and $G_0' \neq K_{2,a}$, $|E(G_0')| \leq 2|V(G_0')| - 5$. By (27) and $|S_1| + |M_0| = 4$,

$$\begin{aligned} 2|V(G_0')| - 5 \geq |E(G_0')| &\geq 3|V(G_0')| + 8 - 4|S_1| - 6|M_0|; \\ 4(|S_1| + |M_0|) + 2|M_0| = 4|S_1| + 6|M_0| &\geq |V(G_0')| + 13; \\ 16 + 2|M_0| &\geq |V(G_0')| + 13; \\ 3 + 2|M_0| &\geq |V(G_0')|. \end{aligned} \tag{28}$$

Since $|S_1| \geq 1$, $|M_0| \leq 3$. By (28), $|V(G_0')| \leq 9$. By Theorem 2.3(b), $|D_2(G_0')| \geq 3$. Since $D_2(G_0') \subseteq S_1$, $|S_1| \geq 3$ and so $|M_0| \leq 1$. By (28), $|V(G_0')| \leq 5$. By Theorem 2.3(a), $G_0' = K_{2,3}$, a contradiction.

Case 2. $G_0' = K_{2,a}$ with $2 \leq a \leq p = 4$.

Since G_0' does not have an SCT, $G_0' = K_{2,3}$. Since $D_2(G_0') \subseteq S_1$, $3 \leq |S_1| \leq 4$. For $v \in S_1$, let $\Gamma(v)$ be the preimage of v in G. Then $|E(G)| = |E(K_{2,3})| + \sum_{v\in S_1} |E(\Gamma(v))| = 6 + \sum_{v\in S_1} |E(\Gamma(v))|$.

If $|S_1| = 4$, then let $S_1 = D_2(G_0') \cup \{u\}$ where $d_{G_0'}(u) = 3$. By Lemma 3.1, $\sigma_t(H) \geq \frac{tn}{4}$ $(1 \leq t \leq 4)$, $|E(\Gamma(v))| \geq |V(\Gamma(v))| - 1$ and $n = |E(G)|$,

$$\begin{aligned} |S_1|\frac{\sigma_t(H) + 2t}{t} &\leq \sum_{v\in S_1}(d_{G_0'}(v) + |V(\Gamma(v))|) \leq \sum_{v\in D_2(G_0')\cup\{u\}} d_{G_0'}(v) + \sum_{v\in S_1}(|E(\Gamma(v))| + 1); \\ n + 8 &\leq 9 + (|E(G)| - 6) + 4 = n + 7, \end{aligned}$$

a contradiction. This shows that $G_0' = K_{2,3}$ with $|S_1| = 4$ is impossible.

If $|S_1| = 3$, then $S_1 = D_2(K_{2,3})$. Let $S_1 = \{v_1, v_2, v_3\}$. To prove $cl(H) = L(G) \in \mathcal{Q}_{2,3}(s_1, s_2, s_3, n)$, we only need to show that for each $v_i \in S_1$, $\Gamma(v_i) = K_{1,s}$ for some $s \geq 1$.

By way of contradiction, we assume that $\Gamma(v_1) \neq K_{1,s}$. Let $e_a = v_1y_1$ and $e_b = v_1y_2$ be the two edges in G_0' incident with v_1 where y_i is a degree 3 vertex in $G_0' = K_{2,3}$ and $d_G(y_i) = d_{G_0'}(y_i) = 3$ $(i = 1, 2)$. Then there are two vertices x_1 and x_2 in $V(\Gamma(v_1))$ such that $x_1y_1 = e_a$ and $x_2y_2 = e_b$ in G.

Claim 1. $\Gamma(v_1)$ contains an edge that is adjacent to at most one of the edges in $\{e_a, e_b\}$.

By $|E(\Gamma(v_1))| \geq 1$, $\Gamma(v_1) \neq K_{1,s}$ and G is an essentially 2-edge-connected K_3-free graph with $\sigma_2(G) \geq 5$, if $x_1 = x_2$, then $\Gamma(v_1)$ contains a cycle C of length at least 4 and so C has an edge that is not adjacent to either edge in $\{e_a, e_b\}$; if $x_1 \neq x_2$, $\Gamma(v_1)$ has an edge that is adjacent to at most one of the edges $\{e_a, e_b\}$. The Claim is proved.

With Claim 1, we may let $e_y = xy$ be such an edge in $\Gamma(v_1)$ that is not adjacent to e_b. Let $e_j = w_jz_j$ be an edge in $E(\Gamma(v_j))$ $(j = 2, 3)$. Then $M_a = \{e_y, e_b, e_2, e_3\}$ is a matching in G.

For $e_b = x_2y_2$, $d_G(x_2)+d_G(y_2) = |E_G(x_2)|+3$. For $e_y = xy$, since G is K_3-free, $|E_G(x)\cap E_G(y)| = 1$ and $|(E_G(x) \cup E_G(y)) \cap E_G(x_2)| \leq 1$, and $E_G(x) \cup E_G(y)) \cup E_G(x_2) \subseteq E(\Gamma(v_1)) \cup \{e_a, e_b\}$. Thus,

$$\begin{aligned} |E_G(x)| + |E_G(y)| + |E_G(x_2)| &= |E_G(x) \cup E_G(y) \cup E_G(x_2)| + |E_G(x) \cap E_G(y)| \\ &\quad +|(E_G(x) \cup E_G(y)) \cap E_G(x_2)| \\ &\leq |E(\Gamma(v_1))| + |\{e_a, e_b\}| + 2 = |E(\Gamma(v_1))| + 4. \end{aligned}$$

Hence,

$$(d_G(x) + d_G(y)) + (d_G(x_2) + d_G(y_2)) = |E_G(x)| + |E_G(y)| + |E_G(x_2)| + 3 \le |E(\Gamma(v_1))| + 7. \quad (29)$$

Since G is K_3-free, $E_G(w_j) \cap E_G(z_j) = \{w_j z_j\}$ and $E_G(w_j) \cup E_G(z_j) \le E(\Gamma(v_j)) \cup E_{G_0'}(v_j)$. Since $v_j \in S_1 = D_2(K_{2,3})$, $|E_{G_0'}(v_j)| = 2$. Then

$$|E_G(w_j)| + |E_G(z_j)| = |E_G(w_j) \cup E_G(z_j)| + |E_G(w_j) \cap E_G(z_j)| \le |E(\Gamma(v_j))| + 3. \quad (30)$$

Thus,

$$\sum_{j=2}^{3}(d_G(w_j) + d_G(z_j)) \le \sum_{j=2}^{3}(|E_G(w_j)| + |E_G(z_j)|) \le |E(\Gamma(v_2))| + |E(\Gamma(v_3))| + 6. \quad (31)$$

By Lemma 3.1 with $\sigma_t(H) \ge \frac{tn}{4}$ and $|M_a| = 4$, by (29), (30), (31) and $|E(G)| = 6 + \sum_{i=1}^{3} |E(\Gamma(v_i))|$,

$$\begin{aligned} |M_a|\frac{\sigma_t(H) + 2t}{t} &\le (d_G(x) + d_G(y)) + (d_G(x_2) + d_G(y_2)) + \sum_{j=2}^{3}(d_G(w_j) + d_G(z_j)); \\ n + 8 &\le |E(\Gamma(v_1))| + 7 + |E(\Gamma(v_2))| + |E(\Gamma(v_3))| + 6 = |E(G)| - 6 + 13 = n + 7, \end{aligned}$$

a contradiction. The proof is completed. □

To prove Theorem 1.10, we need the following theorem:

Theorem 6.1. *(Chen et al. [8]). Let G be a 3-edge-connected graph and let $S \subseteq V(G)$ be a vertex subset with $|S| \le 12$. Then either G has a closed trail C such that $S \subseteq V(C)$, or G can be contracted to P in such a way that the preimage of each vertex of P contains at least one vertex in S.*

Proof of Theorem 1.10. Suppose that H is not Hamiltonian. Let G be the preimage of $cl(H) = L(G)$. Then G is essentially 3-edge-connected. By Theorem 1.1, G does not have a DCT. Let S_0, S_1, S_2, M_0 and U_0 be the sets defined before, where S_0 is the set of all the nontrivial vertices of G_0'. By Theorem 2.4, $\kappa'(G_0') \ge 3$ and G_0' dose not have a DCT containing S_0. Hence, $G_0' \ne K_{2,a}$.

(a) This is a special case of Theorem 1.8 with $k = 3$, $p = 10$, $1 \le t \le 10$ and $\epsilon = 5$. By Lemma 5.1, since $\delta(H) \ge 24 = 3p - 6$, $M_0 = \emptyset$, $S_2 = \emptyset$ and $|S_1| \le p = 10$. Thus, $S_0 = S_1$ and $U_0 = V(G_0') - S_0$.

If $|S_0| \le 9$, then by Theorem 6.1, G_0' has a closed trail C such that $S_0 \subseteq C$. Since U_0 is an independent set, C is a DCT in G_0' containing S_0, a contradiction.

Thus, $|S_0| = 10$. By Lemma 5.1(b), $|V(G_0')| \le 2p - 5 - \epsilon = 10$. By Theorem 2.3(c), $G_0' = P$ and so $S_0 = V(G_0')$. Let $V(G_0') = \{v_1, v_2, \cdots, v_{10}\}$. Let $\Gamma(v_i)$ be the preimage of v_i in G. We assume that

$$|V(\Gamma(v_1))| \le |V(\Gamma(v_2))| \le \cdots \le |V(\Gamma(v_{10}))|. \quad (32)$$

By Lemma 3.1(a), $d_{G'_0}(v) = 3$ for any $v \in V(G'_0)$, $|V(G'_0)| = 10$ and $\sigma_t(H) \geq \frac{t(n+5)}{10}$,

$$\begin{aligned}\sum_{v\in V(G'_0)} |V(\Gamma(v))| + 3|V(G'_0)| &= \sum_{v\in V(G'_0)} \Big(|V(\Gamma(v))| + d_{G'_0}(v)\Big) \geq |V(G'_0)|\frac{\sigma_t(H)+2t}{t} \geq n+25;\\ \sum_{v\in V(G'_0)} |V(\Gamma(v))| &\geq \frac{10\sigma_t(H)}{t} - 10 = (n+5) - 10 = n-5. \quad (33)\end{aligned}$$

Since $|E(\Gamma(v_i))| \geq |V(\Gamma(v_i)| - 1$, by (33), and by $n = |E(G)|$ and $|E(G'_0)| = |E(P)| = 15$,

$$\begin{aligned}n = |E(G)| &= |E(G'_0)| + \sum_{i=1}^{10} |E(\Gamma(v_i))| \geq 15 + \sum_{i=1}^{10}(|V(\Gamma(v_i))| - 1)\\ &\geq 5 + \sum_{i=1}^{10} |V(\Gamma(v_i))| = 5 + (n-5) = n.\end{aligned}$$

Thus, the equalities of (32), (33), and $|E(\Gamma(v_i))| = |V(\Gamma(v_i))| - 1$ must hold. Hence, $\Gamma(v_i)$ is a tree with $|E(\Gamma(v_i))| = |V(\Gamma(v_i))| - 1 = \frac{n-15}{10}$. Since G is essentially 3-edge-connected, $\Gamma(v_i) = K_{1,\frac{n-15}{10}}$. Theorem 1.10(a) is proved.

(b) This is a special case of Theorem 1.8 with $k = 3$, $p = t = 13$ and $\epsilon = 6$. With $\delta(H) \geq 33 = 3p-6$, by Lemma 5.1, $M_0 = \emptyset$, $S_2 = \emptyset$ and $|S_1| \leq p = 13$. Hence, $S_0 = S_1$ and $U_0 = V(G'_0) - S_0$.
Case 1. $|S_0| = |S_1| \leq 12$. Then by Theorem 6.1, we have two subcases:
Subcase (i). G'_0 has a closed trail C such that $S_0 \subseteq C$.

Then C is a DCT in G'_0 that contains all the nontrivial vertices, a contradiction.
Subcase (ii). G'_0 can be contracted to P such that the preimage of each vertex of P contains at least one vertex in S_0. Thus, $G \in \mathcal{P}(n,1)$ and so $cl(H) \in \mathcal{Q}_P(n,1)$. Theorem 1.10 is proved for this case.

Case 2. $|S_0| = |S_1| = p = 13$. By Lemma 5.1, $13 \leq |V(G'_0)| \leq 2p - 5 - \epsilon = 15$ and

$$|E(G'_0)| \geq \epsilon + p + \sum_{v\in U_0} d_{G'_0}(v) = 19 + 3|U_0|. \quad (34)$$

If $13 \leq |V(G'_0)| \leq 14$, then by Theorem 2.3(c). $G'_0 = P_{14}$. Then $|U_0| = 1$. By (34), $|E(G'_0)| \geq 22$, contrary to that $|E(G'_0)| = |E(P_{14})| = 21$.

If $|V(G_0)| = 15$, then $|U_0| = 2$. By (34) $|E(G'_0)| \geq 25$. By Theorem 2.3(d), $V(G'_0) = D_3(G'_0) \cup D_4(G'_0)$ with $|D_4(G'_0)| = 3$. Then $|E(G'_0)| = 24$, a contradiction. Thus, $|S_0| = 13$ is impossible. □

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