Spanning Eulerian subgraphs and Catlin's reduced graphs

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Spanning Eulerian subgraphs and Catlin’s reduced graphs

Wei-Guo Chen∗, Zhi-Hong Chen†

Abstract

A graph $G$ is collapsible if for every even subset $R \subseteq V(G)$, there is a spanning connected subgraph $H_R$ of $G$ whose set of odd degree vertices is $R$. A graph is reduced if it has no nontrivial collapsible subgraphs. Catlin [4] showed that the existence of spanning Eulerian subgraphs in a graph $G$ can be determined by the reduced graph obtained from $G$ by contracting all the collapsible subgraphs of $G$. In this paper, we present a result on 3-edge-connected reduced graphs of small orders. Then, we prove that a 3-edge-connected graph $G$ of order $n$ either has a spanning Eulerian subgraph or can be contracted to the Petersen graph if $G$ satisfies one of the following:

(i) $d(u) + d(v) > 2(n/15 − 1)$ for any $uv \not\in E(G)$ and $n$ is large;
(ii) the size of a maximum matching in $G$ is at most 6;
(iii) the independence number of $G$ is at most 5.

These are improvements of prior results in [16], [18], [24] and [25].

1. Introduction

We shall use the notation of Bondy and Murty [3], except when otherwise stated. Graphs considered in this paper are finite and loopless, but multiple edges are allowed. The graph of order 2 and size 2 is called a 2-cycle and denoted by $C_2$. For a graph $G$, $\kappa'(G)$ and $d_G(v)$ (or $d(v)$) denote the edge-connectivity of $G$ and the degree of a vertex $v$ in $G$, respectively. The set of vertices of degree $i$ in $G$ is denoted by $D_i(G)$. The maximum cardinality of an independent set of vertices in $G$ is denoted by $\alpha(G)$. The size of a maximum matching in $G$ is denoted by $\alpha'(G)$. Let $O(G)$ be the set of vertices of odd degree in $G$. A connected graph $G$ is Eulerian if $O(G) = \emptyset$. An Eulerian subgraph $H$ in $G$ is a spanning Eulerian subgraph if $V(H) = V(G)$. A graph is supereulerian if it has a spanning Eulerian subgraph. A graph $G$ is collapsible if for any even subset $R \subseteq V(G)$ or $R = \emptyset$,

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$G$ has a spanning connected subgraph $H_R$ with $O(H_R) = R$. Examples of collapsible graphs include $K_n$ and $K_{n,n} - e$ ($n \geq 3$).

Throughout this paper, we use $P$ for the Petersen graph and use $P_{14}$ and $P_{16}$ for the graphs defined in Figure 1.1. $P$, $P_{14}$ and $P_{16}$ are 3-edge-connected non-supereulerian graphs.

\begin{figure}[h]
\centering
\begin{subfigure}{.3\textwidth}
  \centering
  \includegraphics[width=\textwidth]{P14.png}
  \caption{P14}
\end{subfigure}
\hfill
\begin{subfigure}{.3\textwidth}
  \centering
  \includegraphics[width=\textwidth]{P16.png}
  \caption{P16}
\end{subfigure}
\caption{Figure 1.1}
\end{figure}

For $X \subseteq E(G)$, the contraction $G/X$ is the graph obtained from $G$ by identifying the two ends of each edge $e \in X$ and deleting the resulting loops. If $H$ is a subgraph of $G$, then we write $G/H$ for $G/E(H)$. If $H$ is connected, then we use $v_H$ denote the vertex in $G/H$ to which $H$ is contracted and $v_H$ is called the contraction image of $H$. Thus, we regard $V(G/H) = (V(G) - V(H)) \cup \{v_H\}$ and $E(G/H) = E(G) - E(H)$.

**Catlin’s reduction method:** In the study of graphs with spanning Eulerian subgraphs and other related graph theory problems such as Hamiltonian line graph and double cycle cover problems [7, 8, 9], Catlin [4] developed a reduction technique by contracting collapsible subgraphs. Catlin [4] showed that every graph $G$ has a unique collection of pairwise disjoint maximal collapsible subgraphs $H_1, H_2, \cdots, H_c$ such that $V(G) = \bigcup_{i=1}^{c} V(H_i)$.

The reduction of $G$ is a graph obtained from $G$ by contracting each $H_i$ into a single vertex $v_i$ ($1 \leq i \leq c$) and is denoted by $G''$. For a vertex $v \in V(G'')$, there is a unique maximal collapsible subgraph $H(v)$ in $G$ such that $v$ is the contraction image of $H(v)$ and $H(v)$ is the preimage of $v$. We regard $K_1$ as a collapsible and supereulerian graph, and having $\kappa'(K_1) = \infty$. A vertex $v \in V(G'')$ is trivial if $v$ is the contraction image of $K_1$. A graph is called Catlin’s reduced or reduced if $G = G''$. We use $\mathcal{CL}$ and $\mathcal{SL}$ to denote the families of collapsible graphs and supereulerian graphs, respectively. Thus, $\mathcal{CL} \subset \mathcal{SL}$. By the definition of contraction, $\kappa'(G'') \geq \kappa'(G)$.

Theorem A below shows the importance of Catlin’s reduction method.

**Theorem A** (Catlin [4]). Let $G$ be a connected graph. Let $G'$ be the reduction of $G$. Let $H$ be a collapsible subgraph of $G$. Then each of the following holds:

(a) $G \in \mathcal{CL}$ if and only if $G/H \in \mathcal{CL}$. Thus, $G \in \mathcal{CL}$ if and only if $G' = K_1$.

(b) $G \in \mathcal{SL}$ if and only if $G/H \in \mathcal{SL}$. Thus, $G \in \mathcal{SL}$ if and only if $G' \in \mathcal{SL}$.
Knowing the structures of reduced graphs of small orders is very important when using the reduction method. The following theorem has been used by many:

**Theorem B** (Chen and Lai [14, 18]). Let $G$ be a connected graph with at most 11 vertices and $\delta(G) \geq 3$. Then either $G \in \mathcal{CL}$ or $G' \in \{K_2, P\}$.

Catlin had a conjecture on reduced graph of order at most 17.

**Conjecture 1** (Catlin [9]). Any 3-edge-connected simple graph of order at most 17 is either supereulerian or is contractible to the Petersen graph.

Several conjectures (see [12, 9, 22]) are extended from Conjecture 1. In this paper, we prove the following theorem, a progress toward solving Conjecture 1.

**Theorem 1.1.** Let $G$ be a 3-edge-connected graph with at most 15 vertices. Let $G'$ be the reduction of $G$. Then each of the following holds:

(a) if $|V(G)| \leq 13$, then either $G \in \mathcal{SL}$ or $G' = P$;
(b) if $|V(G)| \leq 14$, then either $G \in \mathcal{SL}$ or $G' \in \{P, P_{14}\}$;
(c) if $|V(G)| = 15$, $G \not\in \mathcal{SL}$ and $G' \not\in \{P, P_{14}\}$, then $G = G'$ and $G$ is a 2-connected and essentially 4-edge-connected reduced graph with girth at least 5 and $V(G) = D_3(G) \cup D_4(G)$ where $D_4(G)$ is an independent set and $|D_4(G)| = 3$.

Results like Theorem 1.1 play an important role in Catlin’s reduction method, since using Catlin’s reduction method, many problems related to the existence of spanning Eulerian subgraphs can be reduced to the same or similar problems of graphs with very few vertices. Using Theorem 1.1, we obtain the best possible conditions on $\alpha'(G)$ and $\alpha(G)$ for a graph $G$ to be supereulerian. The $\alpha'(G)$ case is better than a conjecture in [25].

**Theorem 1.2.** Let $G$ be a 3-edge-connected simple graph. Let $G'$ be the reduction of $G$. Then each of the following holds:

(a) if $\alpha'(G) \leq 6$, then either $G \in \mathcal{SL}$ or $G' = P$;
(b) if $\alpha(G) \leq 5$, then either $G \in \mathcal{SL}$ or $G' = P$.

Next, we prove the following:

**Theorem 1.3.** Let $G$ be a 3-edge-connected graph of order $n$ and with girth $g$, where $g \in \{3, 4\}$. If $n$ is large enough and

$$d(u) + d(v) > \frac{2}{g-2} \left( \frac{n}{15} - 4 + g \right)$$

for any $uv \not\in E(G)$ (1)

then either $G \in \mathcal{SL}$ or $G' \in \{P, P_{14}\}$. Thus, either $G \in \mathcal{SL}$ or $G$ can be contracted to $P$. 

2. Prior results and $\pi$-reduction method

For a graph $G$, we use $F(G)$ for the minimum number of extra edges that must be added to $G$ to obtain a spanning supergraph having two edge-disjoint spanning trees.

A number of results on reduced graphs are summarized in the following.

**Theorem C.** Let $G$ be a connected reduced graph. Then each of the following holds:

(a) (Catlin [4]). $G$ is $K_3$-free with $\delta(G) \leq 3$ and any subgraph $H$ of $G$ is reduced. Furthermore, $|E(H)| \leq 2|V(H)| - 4$ unless $H \in \{K_1, K_2\}$;

(b) (Catlin [5]). $F(G) = 2|V(G)| - |E(G)| - 2$;

(c) (Catlin et al. [11]). If $F(G) \leq 2$, then $G \in \{K_1, K_2, K_{2,t} \ (t \geq 1)\}$.

(d) (Chen and Lai [19]). If $\delta(G) \geq 3$, then $\alpha'(G) \geq (|V(G)| + 4)/3$.

(e) (Chen [16]). If $\alpha(G) \geq 4$, then $\frac{\delta(G)\alpha(G) + 4}{2} \leq |V(G)| \leq 4\alpha(G) - 5$.

Let $G$ be a graph containing an induced 4-cycle $uwzu$ and let $E = \{uw, vz, zw, wu\}$. Denote by $G/\pi$ the graph obtained from $G - E$ by identifying $u$ and $z$ to form a vertex $x$, and by identifying $v$ and $w$ to form a vertex $y$, and by adding an edge $e_\pi = xy$. The way to obtain $G/\pi$ from $G$ is called the $\pi$-reduction method [5].

\[ K_{3,3} - e \]

\[ \Theta \]

\[ G \]

\[ G/\pi \]

Figure 2.1

Define $\Theta$ as a graph shown in Figure 2.1.

**Theorem D** (Catlin [5]). Let $G$ be a connected graph and let $G/\pi$ be the graph defined above, then each of the following holds:

(a) If $G/\pi \in CL$, then $G \in CL$;

(b) If $G/\pi \in SL$ then $G \in SL$;

(c) If $G$ is $K_3$-free and $G/\pi$ contains $\Theta$ as subgraph, then $G$ has $K_{3,3} - e$ as a collapsible subgraph.

**Notation**: Let $s_{1,2}, s_{2,3}, s_{3,1}, m, l$ and $t$ be natural numbers with $t \geq 2$ and $m, l \geq 1$. Let $\Phi_a \cong K_{1,3}$ with center $a$ and ends $a_1, a_2, a_3$. Define $K_{1,3}(s_{1,2}, s_{2,3}, s_{3,1})$ to be the graph obtained from $\Phi_a$ by adding $s_{i,j}$ vertices
with neighbors \( \{a_i, a_j\} \) (1 ≤ \( i \neq j \) ≤ 3). Note that \( K_{1,3}(1,1,1) \) is the 3-cube minus a vertex. For graphs \( K_{2,1}' \), \( S(m,l) \), \( J(m,l) \), and \( J'(m,l) \), see the Figure 2.2 below. They are reduced graphs.

\[
\begin{align*}
\text{Figure 2.2}
\end{align*}
\]

Let \( F = \{K_1, K_{2,1}', K_{2,1}, K_{1,3}(s', s'', s'''), S(m,l), J(m,l), J'(m,l), P\} \).

Some prior results on reduced graphs of small orders are given in the following theorem:

**Theorem E.** Let \( G \) be a simple connected graph of order \( n \).

(a) (Chen [14]). If \( n \leq 7 \), \( \kappa'(G) \geq 2 \), and \( |D_2(G)| \leq 2 \), then \( G \in \mathcal{C}_L \).

(b) (Catlin [10]). If \( n \leq 8 \), \( \kappa'(G) \geq 2 \) and \( |D_2(G)| \leq 1 \). Then \( G \in \mathcal{C}_L \).

(c) (Chen and Lai [18]). If \( G \) is reduced with \( n \leq 11 \) and \( F(G) \leq 3 \) then either \( G \in F \) or \( G \) is a tree with at most 3 edges.

The following corollary will be needed:

**Corollary 2.1.** Let \( G \) be a connected simple graph of order \( n \) with \( \delta(G) \geq 2 \). Let \( G' \) be the reduction of \( G \).

(a) If \( n \leq 6 \) and \( \delta(G) \geq 2 \) and \( |D_2(G)| \leq 2 \), then \( G \in \mathcal{C}_L \).

(b) If \( n \leq 7 \), \( \delta(G) \geq 2 \) and \( |D_2(G)| \leq 2 \), then \( G' \in \{K_1, K_2\} \).

(c) If \( G \neq K_1 \) is reduced, \( n \leq 7 \), \( \kappa'(G) \geq 2 \) and \( |D_2(G)| = 3 \), then \( G \in \{K_{2,3}, K_{1,3}(1,1,1), J'(1,1)\} \).

(d) If \( n \leq 9 \), \( \delta(G) \geq 2 \) and \( |D_2(G)| \leq 1 \), then \( G' \in \{K_1, K_2, K_{1,2}\} \).

(e) If \( n \leq 9 \), \( \kappa'(G) \geq 2 \) and \( |D_2(G)| \leq 2 \), then \( G' \in \{K_1, K_{2,3}\} \). Furthermore, if \( G \) is \( K_3 \)-free, \( G \in \mathcal{C}_L \).
Proof. For (a) and (b), if \( \kappa'(G) \geq 2 \), then by Theorem E(a), \( G \in \mathcal{CL} \). We may assume that \( \kappa'(G) = 1 \). Let \( e \) be an edge-cut of \( G \). Let \( G_1 \) and \( G_2 \) be the two components of \( G - e \) and \( |V(G_1)| \leq |V(G_2)| \). Since \( \delta(G) \geq 2 \) with \( |D_2(G)| \leq 2 \), \( |V(G_1)| \geq 3 \).

If \( n = 6 \), then \( |V(G_1)| = |V(G_2)| = 3 \). But then \( |D_2(G)| > 2 \), a contradiction. Thus, (a) holds.

If \( n = 7 \), then \( |V(G_1)| = 3 \) and \( |V(G_2)| = 4 \). Since \( |D_2(G)| \leq 2 \), \( G_1 = K_3 \) and \( G_2 = K_4 \). Then \( G' = (G/K_3)/K_4 = K_2 \). Thus, (b) holds.

To prove (c), (d) and (e), we prove the following claim first:

Claim 1. If \( G \) is reduced and \( n + |D_2(G)| \leq 11 \), then \( F(G) \leq 3 \).

Counting the edges in \( G \) we have

\[
|E(G)| = \frac{\sum_{v \in V(G)} d(v)}{2} \geq \frac{2|D_2(G)| + 3(n - |D_2(G)|)}{2} = \frac{3n - |D_2(G)|}{2}. \tag{2}
\]

By Theorem C(b), (2), \( n + |D_2(G)| \leq 11 \),

\[
F(G) = 2|V(G)| - |E(G)| - 2 \leq 2n - \frac{3n - |D_2(G)|}{2} - 2 \leq \frac{7}{2}.
\]

Since \( F(G) \) is an integer, \( F(G) \leq 3 \). Claim 1 is proved.

For (c), since \( G \) is reduced with \( n \leq 7 \) and \( |D_2(G)| = 3 \), \( n + |D_2(G)| \leq 10 \). By Claim 1, \( F(G) \leq 3 \). By Theorem E(c) and \( G \) is not a tree since \( \delta(G) \geq 2 \), \( G \in \mathcal{F} \). Except for the graphs in \( \{K_{2,3}, K_{1,3}(1,1,1), J'(1,1,1)\} \), other graphs \( G \) in \( \mathcal{F} \) either have \( |V(G)| > 7 \), or \( |D_2(G)| \neq 3 \) or \( \kappa'(G) < 2 \). Corollary 2.1(c) is proved.

For (d) and (e), we only need to consider \( 8 \leq n \leq 9 \).

If \( G \) is reduced, then since \( |D_2(G)| \leq 2 \), \( n + |D_2(G)| \leq 11 \). By Claim 1, \( F(G) \leq 3 \). By Theorem E(c) and \( G \) is not a tree, \( G \in \mathcal{F} \). However, each graph in \( \mathcal{F} \) has at least three vertices of degree 2, contrary to the fact that \( |D_2(G)| \leq 2 \). Thus, \( G \) cannot be reduced.

Let \( H \) be a nontrivial maximal collapsible subgraph of \( G \). Let \( v_H \) be the contraction image of \( H \) in \( G/H \). Then \( |V(H)| \geq 3 \) and so \( |V(G/H)| \leq 7 \). If \( V(H) \cap D_2(G) \neq \emptyset \), then \( |D_2(G/H)| \leq |D_2(G)| \leq 2 \). By (b) proved above, \( (G/H)' \in \{K_1, K_2\} \) and so \( G' \in \{K_1, K_2\} \). Thus, (d) is proved for this case. For (e), since \( \kappa'(G) \geq 2 \), \( G' \neq K_2 \). Thus, \( G' = K_1 \). (e) is proved for this case too.

In the following, we assume that \( V(H) \cap D_2(G) = \emptyset \).

Case 1. \( |D_2(G)| \leq 1 \) and \( \delta(G) \geq 2 \) as stated in (d).

If \( d_{G/H}(v_H) \geq 2 \), then since \( |D_2(G)| \leq 1 \), \( |D_2(G/H)| \leq 2 \). Then since \( |V(G/H)| \leq 7 \) with \( |D_2(G/H)| \leq 2 \), by (a) and (b) above, either \( G/H \in \mathcal{CL} \) and so by Theorem A \( G \in \mathcal{CL} \), or the reduction of \( G/H = K_2 \), and so \( G' = K_2 \). Hence, (d) is proved for this case.
If $d_{G/H}(v_H) = 1$, let $e = uv_H$ be the edge incident with $v_H$, which is an edge-cut of $G$. Let $G_1$ be the component of $G - e$ containing $u$. Then $H$ is the other component of $G - e$. Since $V(H) \cap D_2(G) = \emptyset$, $|V(H)| \geq 4$.

If $d(u) > 2$, then $|V(G_1)| \leq 5$ with $\delta(G_1) \geq 2$ and $|D_2(G_1)| \leq 2$, and so by (a), $G_1 \notin \mathcal{CL}$. Thus $G' = G/(H \cup G_1) = K_2$.

If $d(u) = 2$, then $d_{G_1}(u) = 1$. Since $|D_2(G)| = \{u\}$, $d_{G_1}(v) \geq 3$ for $v \in V(G_1) - \{u\}$. Let $H_1 = G_1 - u$. Then $|V(H_1)| \leq 4$ with $|D_2(H_1)| = 1$. By (a) again, $H_1 \in \mathcal{CL}$, thus $G/(H \cup H_1) = K_{1,2}$. Corollary 2.1(d) is proved.

**Case 2.** $|D_2(G)| \leq 2$ and $\kappa'(G) \geq 2$ as stated in (e).

We only need to consider the case that $|D_2(G)| = 2$. Let $G_0 = G/H$. Then $|V(G_0)| \leq 7$ and $|D_2(G_0)| \leq 3$. If $|D_2(G_0)| \leq 2$, then by (b) proved above, $G_0' \in \{K_1, K_2\}$. Since $\kappa'(G_0) \geq \kappa'(G) \geq 2$, $G_0' \neq K_2$. Thus, $G_0' = K_1$ and so $G' = K_1$. We are done in this case.

If $|D_2(G_0)| = 3$, then $d_{G_0}(v_0) = 2$. Therefore, there are only two edges from $G - E(H)$ adjacent to vertices in $V(H)$. Since $V(H) \cap D_2(G) = \emptyset$, $|V(H)| \geq 4$, and so $|V(G_0)| = |V(G/H)| \leq 6$.

If $G_0$ is reduced, then since $|D_2(G_0)| = 3$, $\kappa'(G_0) \geq 2$ and $|V(G_0)| \leq 6$, by (c) above, $G_0 = K_{2,3}$ and so $G' = G_0 = K_{2,3}$. (e) is proved for this case.

If $G_0$ is not reduced, let $H_1$ be a nontrivial maximal collapsible subgraph of $G_0$. Let $G_2 = G_0/H_1$ and let $v_1$ be the contraction image of $H_1$. Since $|V(G_0)| \leq 7$, $|V(G_2)| \leq 4$ if $d_{G_2}(v_1) \geq 3$; or $|V(G_2)| \leq 3$ if $d_{G_2}(v_1) = 2$.

Then $G_2$ is collapsible. Thus, $G' = G_1' = K_1$. Thus, $G' \in \{K_1, K_{2,3}\}$.

If $G$ is $K_3$-free, then any non-trivial collapsible subgraph $H$ of $G$ has order at least 6. Thus, $|V(G/H)| \leq 4$ which implies that $G' \neq K_{2,3}$ and so $G \in \mathcal{CL}$. Corollary 2.1(e) holds.

This completes the proof of Corollary 2.1. □.

**Lemma 2.2.** Let $G$ be a simple and $K_3$-free connected graph of order $n$ where $n \leq 15$ and $|D_2(G)| \leq 2$. Let $H_0 = uvzwu$ be a induced 4-cycle in $G$. Let $G/\pi$ be the graph obtained from $G$ as defined by the $\pi$-reduction method. Then each of the following holds.

(a) If $G/\pi = P$, then $G \in \mathcal{SL}$.

(b) If $\kappa'(G) \geq 2$, $\kappa'(G/\pi) = 1$ and $|D_2(G)| \leq 1$, then $G$ is not reduced.

**Proof.** (a) Since $G/\pi = P$ is a 3-regular graph of order 10, by the definition of $G/\pi$, $|V(G)| = 12$ and $D_2(G) \subseteq V(H_0)$. Then $G$ is one of the graphs in Figure 2.3 (up to isomorphic). As we can see $G_a$ has a hamiltonian cycle $v_1v_2v_5v_3v_7v_6v_4v_8v_1$; $G_b$ (and $G_e$) has a spanning closed trail: $v_1v_6v_7v_4uvzwuv_2v_3v_5v_8v_1$. Thus, $G \in \mathcal{SL}$. 
(b) Since \(\kappa'(G) \geq 2\), \(\epsilon_v\) must be an edge-cut of \(\kappa'(G/\pi)\). Since \(|D_2(G)| \leq 1\), \(G - E(H_0)\) has two non-trivial components, say \(G_1\) and \(G_2\), where \(u, z \in V(G_1)\) and \(v, w \in V(G_2)\) with \(1 < |V(G_1)| \leq |V(G_2)|\). Since \(|V(G)| \leq 15\), \(|V(G_1)| \leq 7\). Since \(|D_2(G)| \leq 1\) and \(\kappa'(G) \geq 2\), \(G_1\) has a 2-edge-connected subgraph \(H_1\) with \(|D_2(H_1)| \leq 2\). By Theorem E(a), \(H_1 \in \mathcal{CL}\), and so \(G\) is not reduced. \(\square\)

**Corollary 2.3.** Let \(G\) be a simple graph of order \(n\) with \(\kappa'(G) \geq 2\).

(a) If \(n \leq 10\) and \(|D_2(G)| \leq 1\), then either \(G \in \mathcal{CL}\) or \(G = P\);

(b) If \(n = 12\), \(|D_2(G)| = 1\) and \(G\) is essentially 3-edge-connected, then either \(G \in \mathcal{SC}\) or \(G\) has a maximal collapsible subgraph \(K_3\) such that \(G' = G/K_3 = P\).

**Proof.** (a) By Corollary 2.1(d), we only need to consider \(|V(G)| = 10\). If \(|D_2(G)| = 0\), then by \(\kappa'(G) \geq 2\), Corollary 2.2 follows from Theorem B.

Next, we assume \(|D_2(G)| = 1\). Let \(v\) be the only vertex in \(D_2(G)\).

**Case 1.** \(G\) contains a \(K_3\). Let \(H\) be a maximal collapsible graph containing a \(K_3\). Then \(G/H\) is simple with \(\kappa'(G/H) \geq 2\). If \(H = K_3\), then \(|V(G/H)| \leq 8\) and \(|D_2(G/H)| \leq 1\), and so by Theorem E(b), \(G/H \in \mathcal{CL}\). Thus \(G \in \mathcal{CL}\). If \(H \neq K_3\), then \(|V(G/H)| \leq 7\) and \(|D_2(G/H)| \leq 2\) and so by Theorem E(a) and by \(\kappa'(G/H) \geq 2\), \(G/H \in \mathcal{CL}\). Hence \(G \in \mathcal{CL}\).

**Case 2.** \(G\) is \(K_3\)-free. Let \(G_1 = G - v\). Since \(|D_2(G)| \leq 1\), \(|D_2(G_1)| \leq 2\). Hence \(|V(G_1)| \leq 9\) and \(|D_2(G_1)| \leq 2\). If \(\kappa'(G_1) \geq 2\), then by Corollary 2.1(e), \(G_1\) and then \(G\) are in \(\mathcal{CL}\). Thus, we may assume that \(\kappa'(G_1) = 1\).

Suppose that \(G_1\) has a cut-edge \(e'\). Let \(G'_1\) and \(G''_1\) be the two components of \(G_1 - e'\). By \(|D_2(G)| \leq 1\), each of \(G'_1\) and \(G''_1\) contains at least 3 vertices and so each satisfies the hypothesis of Corollary 2.1(a). Thus both \(G'_1\) and \(G''_1\) are in \(\mathcal{CL}\) and so \((G/G'_1)/(G''_1) = K_3\). Hence \(G \in \mathcal{CL}\).

(b) Suppose that \(G \notin \mathcal{SC}\) and so \(G' \neq K_1\). Let \(D_2(G) = \{v\}\).

**Case 1.** \(G\) has a \(K_3\) subgraph. Let \(H\) be a maximum collapsible subgraph containing a \(K_3\). Let \(v_H\) be the contraction image of \(H\) in \(G/H\).
If \( H = K_3 \) then \( d_{G/H}(v_H) \geq 2 \) if \( v \in V(H) \); otherwise, \( d_{G/H}(v_H) \geq 3 \). Thus, \( |V(G/H)| = 10 \) and \( |D_2(G/H)| \leq 1 \). Since \( G' = (G/H)' \neq K_1 \), by (a) above, \( G/H = G/K_3 = P \).

If \( H \neq K_3 \), then \( |V(H)| \geq 4 \), and so \( |V(G/H)| \leq 9 \) and \( |D_2(G/H)| \leq 2 \). By Corollary 2.1(e) and \( G' \neq K_1 \), \( G' = (G/H)' = K_{2,3} \). Since \( |D_2(G)| = 1 \), at least two vertices of degree 2 in \( G' \) are contractions of nontrivial collapsible subgraphs of \( G \). Thus, \( G \) has an essential edge-cut of size 2, contrary to the fact that \( G \) is essentially 3-edge-connected.

**Case 2.** \( G \) is \( K_3 \)-free but has a 4-cycle \( H_0 \). Let \( G/\pi \) be the graph defined by the \( \pi \)-reduction method. Then \( |V(G/\pi)| = 10 \) and \( |D_2(G/\pi)| \leq 1 \).

If \( \kappa'(G/\pi) \geq 2 \), then by (a) above, either \( G/\pi \in \mathcal{CL} \) or \( G/\pi = P \). By Theorem D or Lemma 2.2(a), \( G \in \mathcal{SL} \), a contradiction.

If \( \kappa'(G/\pi) = 1 \), then since \( |D_2(G/\pi)| \leq 1 \), by Lemma 2.2(b), \( G \) is not reduced. Let \( H \) be a nontrivial maximal collapsible subgraph of \( G \). Since \( G \) is \( K_3 \)-free, \( |V(H)| \geq 6 \). Thus, \( |V(G/H)| \leq 7 \) with \( |D_2(G/H)| \leq 2 \) and \( \kappa'(G/H) \geq \kappa'(G) \geq 2 \). By Theorem E(a), \( G/H \in \mathcal{CL} \), a contradiction.

**Case 3.** \( G \) has no 3- and 4-cycles. Since \( |D_2(G)| = 1 \) and \( G \) cannot have 11 vertices of degree 3, \( \Delta(G) \geq 4 \). Let \( z \) be a vertex of degree \( \Delta(G) \). Let \( N(z) = \{y_1, y_2, y_3, y_4, \cdots \} \). Since \( G \) has no 3- or 4-cycles, \( (N(y_i) - z) \cap (N(y_j) - z) = \emptyset \) for any \( i \neq j \) and \( 1 \leq i, j \leq 4 \). Since \( |D_2(G)| = 1 \), at least 3 of \( N(y_i) - z \) has at least 2 vertices. We may assume that \( |N(y_i) - z| \geq 1 \) and \( |N(y_i) - z| \geq 2 \) for \( i = 2, 3, 4 \). Let \( S = \bigcup_{i=1}^{4} (N(y_i) - z) \). Then \( |S| \geq 7 \). Since \( \{z\} \cup N(z) \cup S \subseteq V(G) \), \( 12 = |V(G)| \geq 1 + 4 + 7 = 12 \). Thus, \( \Delta(G) = 4 \), \( |S| = 7 \) and \( G \) has only one vertex of degree 4 which is adjacent to the vertex in \( D_2(G) \), which is \( y_1 \). Furthermore, every vertex in \( S \) has degree 3. Let \( G_S = G[S] \). Then \( d_{G_S}(v) \geq 2 \) for any \( v \in S \). Since \( G \) has no 3- and 4-cycles, \( G_S = C_7 \). Thus, \( G \) must be isomorphic to the graph in Figure 2.4, which has a Hamiltonian cycle: \( zy_1s_1s_2s_5s_4s_3s_6s_7y_4z \), contrary to the fact that \( G \notin \mathcal{SL} \). □

![Figure 2.4](image)

Define \( \mathcal{ST} \) as the set of graphs \( H \) with the property that \( \delta(H) = 2 \) and for any two vertices \( u, v \in D_2(H), H \) has a spanning \((u, v)\)-trail. Note that \( \{K_{2,3}, K_{1,3}(1, 1, 1), J'(1, 1)\} \in \mathcal{ST} \).
Lemma 2.4. Let $H \in ST$. Let $G_1$ be a super-eulerian graph with a vertex $v$ of degree 2 or 3. Let $G$ be a graph obtained from $G_1$ by replacing the vertex $v$ by $H$ with the edges incident with $v$ joining different vertices in $D_2(H)$. Then $G \in SL$.

Proof. Since $G_1 \in SL$, $G_1$ has a spanning closed trail $T : v e_1 v_1 \cdots v_k e_k v$ with $e_1$ and $e_k$ incident with $v$ in $G_1$. Since each edge incident with $v$ in $G_1$ is incident with a vertex in $D_2(H)$ in $G$, let $x_1$ and $x_2$ be the two vertices in $D_2(H)$ incident with $e_1$ and $e_k$, respectively. Since $H$ has a spanning $(x_1, x_2)$-trail, let $T_H$ be a spanning $(x_1, x_2)$-trail in $H$. Therefore, $G[T \cup T_H]$ is a spanning closed trail in $G$. □

To avoid long and complicated case by case arguments, we will use the computer search results obtained by David Pike in [23]. David Pike [23] found all the Non-Hamiltonian Cubic 2-edge-connected graphs of order up to 16: 1 graph of order 10 (the Petersen graph), 1 graph of order 12 (which contains a $K_3$), 6 graphs of order 14 (only $P_{14}$ is $K_3$-free), and 33 graphs of order 16. The completed list of those graphs can be found in [23]. We are only interested in reduced cubic 2-edge-connected graphs. After excluded non-reduced graphs from the list, we have the following:

Theorem F (Pike [23]). If $G$ is a cubic 2-edge-connected Non-Hamiltonian reduced graph of order at most 16, then $G$ can be contracted to the Petersen graph. Further more,

(a) If $G$ is not the Petersen graph, $G$ has girth 4;
(b) If $|V(G)| \leq 12$, then $G$ is the Petersen graph;
(c) If $|V(G)| = 14$, then $G$ is the graph $P_{14}$.

3. Applications of Theorem 1.1

It was proved in [16] that if $G$ is a 3-edge-connected simple graph with $\alpha(G) \leq 4$, then either $G \in SC$ or $G' = P$. We show that this is still true for $\alpha(G) \leq 5$. For maximum edge independence number $\alpha'(G)$, it was proved in [18] that for a 3-edge-connected simple graph $G$ with $\alpha'(G) \leq 5$, either $G \in SC$ or $G' = P$. Not knowing this result, Yan in [25] posted it as a conjecture. Theorem 1.2 is an improvement of these results.

Proof of Theorem 1.2. (a) By Theorem C(d), a connected reduced graph of order $n$ with $\delta(G) \geq 3$ has $\alpha'(G) \geq (n + 4)/3$. Thus, $(n + 4)/3 \leq 6$, and so $n \leq 14$. By Theorem 1.1, either $G \in SC$ or $G' \in \{P, P_{14}\}$. But $\alpha'(P_{14}) = 7$. Theorem 1.2(a) is proved.

(b) By way of contradiction, suppose that $G \notin SC$ and $G' \neq P$. By Theorem C(e) and $\alpha(G) \leq 5$, $|V(G)| \leq 4 \alpha(G) - 5 \leq 15$. Then by Theorem 1.1, $|V(G)| = 14$ or 15. If $|V(G)| = 14$, then $G = P_{14}$ and so $\alpha(G) = \alpha(P_{14}) = 6$, a contradiction.
If \(|V(G)| = 15\), then by Theorem 1.1, \(G\) has girth at least 5 and \(D_4(G)\) is an independent set with \(|D_4(G)| = 3\). Let \(v \in D_4(G)\). Let \(N_G(v) = \{x_1, x_2, x_3, x_4\}\). Since \(D_4(G)\) is independent, \(D_4(G) \cap N_G(v) = \emptyset\), and so \(d(x_i) = 3\) (\(1 \leq i \leq 4\)). Let \(N_G(x_i) - \{v\} = \{y_{i1}, y_{i2}\}\). Since \(G\) has girth at least 5, \((N_G(x_i) - \{x_i\}) \cap (N_G(y_{ij}) - \{v\}) = \emptyset\) (\(i \neq j\)). Let \(S = \{y_{11}, y_{12}, y_{21}, y_{22}, y_{31}, y_{32}, y_{41}, y_{42}\}\). Since \(|V(G)| = 15\), there are two vertices in \(V(G) - (S \cup N_G(v) \cup \{v\})\). Let \(V(G) - (S \cup N_G(v) \cup \{v\}) = \{z_1, z_2\}\). If \(\{z_1, z_2\} \subset D_4(G)\), then \(z_1z_2 \notin E(G)\), and so \(\{z_1, z_2, x_1, x_2, x_3, x_4\}\) is an independent set in \(G\), a contradiction. Thus, at least one of \(\{z_1, z_2\}\) (say \(z_1\)) has degree 3. Then there is a vertex in \(\{x_1, x_2, x_3, x_4\}\) (say \(x_1\)) such that \(z_1\) is not adjacent to any vertices in \(N_G(x_1)\). Therefore, \(\{z_1, y_{11}, y_{12}, x_2, x_3, x_4\}\) is an independent set in \(G\), contrary to \(\alpha(G) \leq 5\). □.

The following theorem was proved by Catlin [6] and Chen [15]:

**Theorem G** (Chen [15]). Let \(p \geq 2\) be an integer. Let \(G\) be a 2-edge-connected simple graph of order \(n\) with girth \(g\), where \(g \in \{3, 4\}\). Let \(G'\) be the reduction of \(G\). If \(n \geq 4(g - 2)p^2\) and

\[
d(u) + d(v) > \frac{2}{g - 2} \left(\frac{n}{p} - 4 + g\right)
\]

for any \(uv \notin E(G)\), then either \(G \in \mathcal{CL}\) or \(G' \neq K_1\) is a graph of order less than \(p\). In particular, either \(G \in \mathcal{SL}\) or \(G' \notin \mathcal{SL}\) with order less than \(p\).

The Dirac degree condition below implies the degree-sum condition (3).

\[
\delta(G) > \frac{1}{g - 2} \left(\frac{n}{p} - 4 + g\right).
\]

The case \(p = 5\) in (4) was conjecture by Bauer [1], of which the case \(g = 3\) was proved by Catlin [4], and the case \(g = 4\) was proved by Lai [20].

The case \(p = 2\) with \(g = 3\) in (3) was proved by Lesniak-Foster and Williamson [21]. The case \(p = 5\) with \(g = 3\) in (3) was proved by Catlin [6], which was conjectured by Benhocine et al. [2].

For 3-edge-connected graphs, Chen [14] proved the case \(p = 10\) with \(g = 3\) in (4), Catlin [6] proved the case \(p = 10\) with \(g = 3\) in (3), and Chen [15] proved the case \(p = 11\) in (3). Li et al [22] proved the following:

**Theorem H** (Li et al. [22]). Let \(G\) be a 3-edge-connected graph of order \(n\). Then each of the following holds:

(a) If \(\delta(G) \geq \frac{n - 31}{12}\) and \(n \geq 61\), then either \(G \in \mathcal{SL}\) or \(G' = P\).

(b) If \(G\) is \(K_3\)-free, \(\delta(G) \geq \frac{n - 25}{24}\) and \(n \geq 97\), then either \(G \in \mathcal{SL}\) or \(G' = P\).

Theorem 1.3 is an improvement of Theorem H.

**Proof of Theorem 1.3.** Theorem 1.3 is the case \(p = 15\) in Theorem G.
Let $G'$ be the reduction of $G$. Suppose that $G$ is not collapsible and so $G' \neq K_1$. Since $\kappa'(G') \geq \kappa'(G) \geq 3$, by Theorem G, $|V(G')| < 15$. By Theorem 1.1, either $G \in \mathcal{S}\mathcal{C}$ or $G' \in \{P, P_{14}\}$. Since $P_{14}$ can be contracted to the Petersen graph by contracting the $K_{2,3}$ into a single vertex, either $G \in \mathcal{S}\mathcal{C}$ or $G$ can be contracted to $P$. Theorem 1.3 is proved. □.

**Corollary 3.3.** Let $G$ be a $3$-edge-connected graph of order $n$. Then when $n$ is large, each of the following holds:

(a) If $\delta(G) > \frac{n}{16} - 1$, then either $G \in \mathcal{S}\mathcal{C}$ or $G' \in \{P, P_{14}\}$.

(b) If $G$ is $K_3$-free and $\delta(G) > \frac{n}{36}$, then either $G \in \mathcal{S}\mathcal{C}$ or $G' \in \{P, P_{14}\}$.

**Remark:** Let $G$ be a graph obtained from $P_{16}$ by replacing each vertex by a $K_{n/16}$. Then $G$ is a $3$-edge-connected graph of order $n$ with $\delta(G) \geq n/16 - 1$. However the reduction of $G$ is $P_{16}$. Thus, the degree condition in Corollary 3.3 cannot be reduced to $\delta(G) \geq n/16 - 1$. But if we relax the conclusions of Theorem 1.3 and Corollary 3.3 from ”the reduction of $G$ is in $\{P, P_{14}\}$” to ”$G$ can be contracted to $P'$”, we have the following conjecture:

**Conjecture 2** (Catlin et al [13]). Let $G$ be a $3$-edge-connected graph of order $n$ with girth $g \in \{3, 4\}$. If $d(u) + d(v) > \frac{2}{g-2} \left( \frac{n}{18} - 4 + g \right)$ for any $uv \notin E(G)$ and $n$ is large, then either $G \in \mathcal{S}\mathcal{C}$ or $G$ can be contracted to $P$.

Let $G$ be a graph obtained from a Blanuša snark by replacing each vertex by a $K_{n/18}$ or $K_{n/36,n/36}$. Then $\delta(G) = \frac{2}{g-2} \left( \frac{n}{18} - 4 + g \right)$. But the reduction of $G$ is the Blanuša snark and cannot be contracted to the Petersen graph. Thus, the degree condition in Conjecture 2 is the best possible.

### 4. An Associated Result

**Theorem 4.1.** Let $G$ be a $3$-edge-connected graph with $|V(G)| \leq 15$. Let $G'$ be the reduction of $G$. Then one of the following holds:

(a) $G \in \mathcal{S}\mathcal{C}$ or

(b) $G' \in \{P, P_{14}\}$, or

(c) $G'$ is a graph satisfying each of the following:

(i) $G'$ is 2-connected, 3-edge-connected and essentially 4-edge-connected;

(ii) $G'$ has girth at least 5;

(iii) $\Delta(G') = \{v \in V(G') \mid d_{G'}(v) \geq 4\}$ is an independent set;

(iv) $\Delta(G') \leq \left\lfloor \frac{|V(G')| - 1}{3} \right\rfloor$. 

Proof. By way of contradiction, suppose that

\[ G \] is a counterexample to (a) and (b) with \(|E(G)|\) minimized. \hfill (5)

Since the reduction method preserves the edge-connectivity, by Theorem B and (5), we may assume that \(G = G'\) and \(12 \leq |V(G)| \leq 15\).

**Claim 1.** \(G\) is 2-connected.

Suppose that \(G\) has a vertex cut \(v\). Let \(H_1\) and \(H_2\) be the two components of \(G - v\). Let \(G_i = G[H_i \cup v]\) (\(i = 1, 2\)). Since \(\kappa'(G) \geq 3\), \(G_i\) is 3-edge-connected. We may assume that \(|V(G_1)| \leq |V(G_2)|\). Then \(|V(G_1)| \leq (|V(G)| + 1)/2 \leq 8\). Thus, by Theorem B, \(G_1 \in \mathcal{CL}\), contrary to the fact that \(G\) is a reduced graph. Claim 1 is proved.

**Claim 2.** \(G\) is essentially 4-edge-connected.

Suppose that \(G\) has an essential edge cut \(X \subseteq E(G)\) with \(|X| = 3\). Let \(G_1\) and \(G_2\) be the two components of \(G - X\) with \(|V(G_1)| \leq |V(G_2)|\). Then \(|V(G_1)| \leq 7\).

Since \(\kappa'(G) \geq 3\) and \(|X| = 3\), \(\kappa'(G_1) \geq 2\). If \(|D_2(G_1)| \leq 2\), then by Theorem E(a), \(G_1\) is not reduced, a contradiction. Thus, the three edges in \(X\) must be incident with three different vertices in \(G_1\) respectively, and \(|D_2(G_1)| = 3\). Since \(\kappa'(G_1) \geq 2\), \(|V(G_1)| \leq 7\) and \(|D_2(G_1)| = 3\), by Corollary 2.1(c), \(G_1 \in \{K_{2,3}, K_{1,3}(1,1,1), J'(1,1)\}\). Therefore, either \(|V(G_1)| = 5\), or \(|V(G_1)| = 7\), and so \(|V(G_2)| = |V(G)| - |V(G_1)| \leq 10\).

Let \(G_0 = G_1G\) with \(v_0\) as the contraction image of \(G_1\). Then \(G_0\) is the graph obtained from \(G_2\) and \(v_0\) by joining the edges in \(X\) from \(G_2\) to \(v_0\), and so \(|V(G_0)| \leq 11\). Since \(\kappa'(G) \geq 3\), \(\kappa'(G_0) \geq 3\). By Theorem B, either \(G_0 \in \mathcal{SL}\) or \(G_0' = P\). If \(G_0 \in \mathcal{SL}\), then since \(G_1 \in \{K_{2,3}, K_{1,3}(1,1,1), J'(1,1)\} \subseteq ST\), by Lemma 2.4, \(G \in \mathcal{SL}\), a contradiction. If \(G_0' = P\), then \(G_1 = K_{2,3}\) and so \(G = P_{14}\), a contradiction. Claim 2 is proved.

**Claim 3.** \(G\) has no 4-cycle.

By way of contradiction, suppose \(G\) has a 4-cycle, say \(H_0 = uvzwu\). Let \(G/\pi\) be the graph defined by the \(\pi\)-reduction method with \(e_\pi = xy\) as the new edge in \(G/\pi\). Since \(\kappa'(G) \geq 3\) and \(12 \leq |V(G)| \leq 15\), by the definition of \(G/\pi\), \(\delta(G/\pi) \geq 3\), \(\kappa'(G/\pi) \geq 1\) and

\[ 10 \leq |V(G/\pi)| = |V(G)| - 2 \leq 13. \]

**Case 1.** \(\kappa'(G/\pi) \geq 3\). By (5), since \(G\) is a minimum counterexample and \(|V(G/\pi)| < |V(G)|\), either \(G/\pi \in \mathcal{SL}\) or \((G/\pi)' = P\). Since \(G \notin \mathcal{SL}\), by Theorem D, \(G/\pi \notin \mathcal{SL}\). Thus, the reduction of \(G/\pi\) is \(P\).

**Case 1A.** \(|V(G/\pi)| = 10\). Then \(G/\pi = P\). By Lemma 2.2 \(G \in \mathcal{SL}\), a contradiction.
Case 1B. $|V(G/\pi)| = 11$. Then since the reduction of $G/\pi$ is $P$, $G/\pi$ contains a cycle $C_2$ of length 2. Since $G$ is $K_3$-free, the $C_2$ in $G/\pi$ is formed by the $\pi$-reduction operation on $G$. By the definition of $G/\pi$ and $\kappa'(G) \geq 3$, $G$ must be one of the two graphs in Figure 4.2. Both are supereulerian ($\Psi_1$ has a spanning closed trail $x_1x_2u_1ux_3x_6x_8wzu_1x_4x_7x_1$, and $\Psi_2$ has a hamiltonian cycle $x_1x_2zv_5x_3x_4x_1wux_4x_7x_1$), a contradiction again.

Case 1C. $|V(G/\pi)| = 12$. Since the reduction of $G/\pi$ is $P$, $G/\pi$ either contains a $K_3$ or two $C_2$ such that $(G/\pi)/K_3 = P$ or $(G/\pi)/(C_2 \cup C_2) = P$.

Case 1C(i). $(G/\pi)/K_3 = P$. If $e_\pi \in E(K_3)$, then since $G$ is $K_3$-free with $\delta(G) \geq 3$, $G$ is a graph with the structure as shown in Figure 4.3, which has an essential 3-edge-cut, contrary to the fact that $G$ is essentially 4-edge-connected.

If $e_\pi \notin E(K_3)$, then $G/\pi$ and $G$ are the graphs as shown in Figure 4.4 below. Graph $G$ in Figure 4.4 has a hamiltonian cycle: $C = x_1x_2x_3x_5x_{10}x_9x_6x_7uvx_4x_8wzx_1$, contrary to (5).
Case 1C(ii). \((G/\pi)/(C_2 \cup C_2) = P\). Since \(G\) is reduced, by Theorem D(c), \(G/\pi\) has no \(\Theta\) as a subgraph. The two \(C_2\) cycles must be incident with edge \(e_3\) in \(G/\pi\) as shown in Figure 4.5. By the definition of \(G/\pi\), graph \(G\) is isomorphic to the graph shown in Figure 4.5.

\[
G/\pi : \quad G:
\]

\[
\begin{array}{c}
\text{Figure 4.5}
\end{array}
\]

The subgraph \(H = x_1x_2x_3x_4x_5x_6x_7x_8x_9x_{10}x_{11}\) is a spanning Eulerian subgraph in \(G\) in Figure 4.5, a contradiction.

Case 1D. \(|V(G/\pi)| = 13\). Since \(G\) is reduced, the reduction of \(G/\pi\) is \(P\) and \(G/\pi\) has no \(\Theta\) as a subgraph, \(G/\pi\) either contains a \(K_3\) and a \(C_2\) such that \((G/\pi)/(K_3 \cup C_2) = P\) or contains a collapsible subgraph \(H\) of order 4 such that \((G/\pi)/H = P\).

Case 1D(i). \(G/\pi\) contains a \(K_3\) and a \(C_2\) such that \((G/\pi)/(K_3 \cup C_2) = P\).

Since \(G\) is reduced, \(K_3\) and \(C_2\) are generated by the \(\pi\)-reduction on \(G\). Since \(\delta(G) \geq 3\) and \(P\) is 3-regular, \(G/\pi\) has one of the two configurations shown in Figure 4.6. Thus, \(G\) is one of the graphs \(G_a\) and \(G_b\) in Figure 4.6.

\[
G = G_a \quad \text{Or} \quad G = G_b
\]

\[
\begin{array}{c}
\text{Figure 4.6}
\end{array}
\]

\(G_a\) has a spanning closed trail: \(x_{11}x_1x_2x_3x_9x_{10}x_6x_7x_8x_{11}uwx_4x_5uxz_{x_11}\); and \(G_b\) has a hamiltonian cycle: \(x_{11}x_8x_9x_3x_2x_7x_6x_{10}x_1uwx_4x_5uxz_{x_11}\). Thus, \(G \in \{G_a, G_b\} \subset \mathcal{S}\), a contradiction. Case 1D(i) is proved.

Case 1D(ii). \(G/\pi\) contains a collapsible subgraph \(H\) of order 4 such that \((G/\pi)/H = P\).

Let \(v_0\) be the contraction image of \(H\) in \(P\). There are exactly three edges incident with \(H\). By Claim 2 above, \(G\) is essentially 4-edge-connected and so \(e_\pi \not\in E(H)\) and \(e_\pi\) is an edge in \(E(P)\) incident with \(v_0\). Let \(E(v_0) = \{e_\pi, e_a, e_b\}\) be the set of three edges in \(P\) incident with \(v_0\), and so the edges
in \(E(v_0)\) are the only three edges joining \(H\) to \((G/\pi)-H\). Since \(|V(H)| = 4\), at least one vertex in \(V(H)\) is not incident with any edges in \(E(v_0)\).

Let \(V(H) = \{u_0, u_1, u_2, x\}\) where \(x\) is incident with \(e_\pi\) and \(u_0\) is not incident with any edges in \(E(v_0)\). Since \(\delta(G/\pi) \geq 3\), either \(\{u_1, u_2, x\} \subseteq N_H(u_0)\), or \(u_0\) is adjacent to only one of the vertices in \(\{u_1, u_2\}\) (say \(u_2\)) and two parallel edges joining \(u_0\) and \(x\) (see Figure 4.7 (II)).

\[
\text{(I)} \quad \text{Or} \quad \text{(II)}
\]

**Figure 4.7**

**Subcase 1D(ii)(I).** \(N_H(u_0) = \{u_1, u_2, x\}\) (see Figure 4.7 (I)).

Since \(G\) is \(K_3\)-free, \(u_1u_2 \notin E(G)\). Therefore, since \(G\) is essentially 4-edge-connected and \(\delta(G/\pi) \geq 3\), \(u_i\) (\(i = 1, 2\)) must be adjacent to \(x\) and incident with an edge in \(E(v_0)\) as shown in Figure 4.8 below.

\[
G/\pi \quad \iff \quad G
\]

**Figure 4.8**

Thus \(G\) is the graph in Figure 4.8 which has a spanning closed trail \(H_1 = x_1x_2x_6x_5ux_1x_3x_4x_7x_8u_2u_0wzx_1\), contrary to (5).

**Subcase 1D(ii)(II).** \(|N_H(u_0) \cap \{u_1, u_2\}| = 1\), and \(u_0\) and \(x\) are joined by a pair of edges.

Since \(|N_H(u_0) \cap \{u_1, u_2\}| = 1\), \(u_0\) is adjacent to only one vertex in \(\{u_1, u_2\}\), say \(u_2\) as shown on Figure 4.7 (II). Since \(G\) is \(K_3\)-free, \(u_2x \notin E(H)\). Since \(d(u_i) \geq 3\) and \(u_1\) cannot be adjacent to both \(u_0\) and \(x\), \(u_1\) must be adjacent to \(x\) and \(u_2\) (see Figure 4.9). But \(G\) has a Hamiltonian cycle: \(x_1x_2x_6x_5ux_1x_3x_4x_7x_8u_2u_0wzx_1\), a contradiction again.

This completes the proof of Claim 3 for the case \(\kappa'(G/\pi) \geq 3\).
Case 2. \( \kappa'(G/\pi) = 2 \).

Since \( \kappa'(G) \geq 3 \), \( e_\pi = xy \) must be in any edge-cuts of size 2 in \( G/\pi \), where \( x \) is the vertex obtained from \( G \) by identifying \( u \) and \( z \) and \( y \) is the vertex obtained by identifying \( v \) and \( w \). Let \( X = \{ e_\pi, e_0 \} \) be an edge cut of size 2 in \( G/\pi \) where \( e_0 = ab \). Let \( X_1 = E(H_0) \cup \{ e_0 \} \). Therefore, \( G - X_1 \) has two components, say \( G_1 \) and \( G_2 \). We may assume that \( u, z, a \in V(G_1) \) and \( v, w, b \in V(G_2) \) and \( |V(G_1)| \leq |V(G_2)| \).

Claim 2A. \( d_{G/\pi}(y) \geq 4 \).

By way of contradiction, suppose that \( d_{G/\pi}(y) \leq 3 \). Then \( e_0 \) with the edges other than \( e_\pi \) incident with \( y \) forms an essential edge-cut with size at most 3, contrary to Claim 2 that \( G \) is essentially 4-edge-connected. Thus, \( d_{G/\pi}(y) \geq 4 \). Claim 2A is proved.

Subcase 1. Every edge in \( G_1 \) is incident with a vertex in \( \{ u, z \} \cup \{ a \} \).

Since \( G \) is \( K_3 \)-free and \( \delta(G) \geq 3 \), either \( G_1 \) has a \( K_{2,3} \) subgraph, or \( G_1 = K_{1,2} \) with \( V(G_1) = \{ u, z, a \} \) where \( d_{G_1}(a) = 2 \).

If \( G_1 \) has a \( K_{2,3} \) subgraph, then \( G \) has a \( K_{3,3} - \varepsilon \) subgraph, contrary to the fact that \( G \) is reduced.

If \( G_1 = K_{1,2} \), then \( G/\pi \) has a \( C_2 \) with \( V(C_2) = \{ x, a \} \). Let \( G_0 = (G/\pi)/C_2 \). Let \( v_e \) be the contraction image of \( C_2 \) in \( G_0 \). Then \( d_{G_0}(v_e) = 2 \) with \( v_e, b \in E(G_0) \). Note that \( d_{G_0}(y) = d_{G/\pi}(y) \geq 4 \). Furthermore, \( G_0 \) is essentially 3-edge-connected. Otherwise, if \( G_0 \) has an essential edge-cut \( X_0 \) with \( |X_0| \leq 2 \), then either \( X_0 \) is an essential edge-cut of \( G \) if \( e_\pi \notin X_0 \), or \( (X_0 - e_\pi) \cup \{ au, az \} \) if \( e_\pi \in X_0 \) is an essential edge-cut of \( G \), contrary to the fact that \( G \) is essentially 4-edge-connected. Since \( G_0 \) is essentially 3-edge-connected with \( |V(G_0)| = 12 \) and \( |D_2(G_0)| = 1 \), by Corollary 2.3(b), either \( G_0 \in SL \) or \( G_0' = P \).

If \( G_0 \in SL \), then by Theorem A, \( G/\pi \in SL \). By Theorem D, \( G \in SL \), a contradiction.
If $G'_{0} = P$, since $|D_{2}(G_{0})| = 1$ and $d_{G_{0}}(y) \geq 4$, $G_{0}$ has a collapsible subgraph $H_{0}$ that contains $y$ and has $|V(H_{0})| \geq 4$ such that $G'_{0} = (G_{0}/H_{0})' = P$. Then $10 = |V(G'_{0})| \leq |V(G_{0}/H_{0})| \leq 9$, a contradiction.

**Subcase 2.** $G_{1}$ has an edge $x_{1}x_{2}$ that is not incident with any vertices of $\{u, z\} \cup \{a\}$.

Since $G$ is $K_{3}$-free, $N_{G}(x_{1}) \cap N_{G}(x_{2}) = \emptyset$ and $N_{G}(x_{1}) \cup N_{G}(x_{2}) \subseteq V(G_{1})$. We have

$$|V(G_{1})| \geq |N_{G}(x_{1})| + |N_{G}(x_{2})| \geq d(x_{1}) + d(x_{2}) \geq 3 + 3 = 6. \quad (6)$$

Since $|V(G)| \leq 15$ and $|V(G_{1})| \leq |V(G_{2})|$, by (6),

$$6 \leq |V(G_{1})| \leq 7 \quad \text{and} \quad |V(G_{2})| \leq 9.$$

Let $H_{1}$ and $H_{2}$ be the two components of $G/\pi - X$. Then $|V(H_{i})| = |V(G_{1})| - 1 = 6$ and $|V(G_{2})| = |V(G_{2})| - 1 \leq 8$, and $D_{1}(H_{i}) \cup D_{2}(H_{i}) \subseteq \{x, y, a, b\}$ ($i = 1, 2$).

If $G/\pi$ is simple, then since $\kappa'(G) \geq 3$, $D_{1}(H_{i}) = \emptyset$, and $\kappa'(H_{i}) \geq 2$ ($i = 1, 2$) and $|D_{2}(H_{i})| \leq 2$. By Corollary 2.1 and $\kappa'(H_{i}) \geq 2$, $H_{i} \in \mathcal{CL}$. Therefore, $(G/\pi)/(H_{1} \cup H_{2}) = C_{2}$ and so $G/\pi \in \mathcal{CL}$. By Theorem D, $G \in \mathcal{CL}$, a contradiction.

If $G/\pi$ is not simple, then $G/\pi$ contains $C_{2}$ cycles formed by the $\pi$-reduction operation on $G$. Since reduced $G$ has no $K_{3,3} - e$, $G/\pi$ has no $\Theta$ as a subgraph. Thus, all the $C_{2}$ cycles are incident with only one end of the vertices of $e_{\pi} = xy$.

We may assume that all the $C_{2}$ cycles incident with $x$. (The case that all the $C_{2}$’s incident with $y$ can be proved in the same way and so omitted). Let $H_{c}$ be the maximal collapsible subgraph in $H_{1}$ containing all the $C_{2}$ cycles. Let $H_{c}^{*} = H_{1}/H_{c}$. Let $v_{c}$ be the contraction image of $H_{c}$. We regard $v_{c} = x$ and $v_{c}y = xy$. Then $|V(H_{c}^{*})| \leq |V(H_{1})| - 1 \leq 5$ and $d_{H_{c}}(v_{c}) \geq 2$. Thus, $D_{1}(H_{c}^{*}) \cup D_{2}(H_{c}^{*}) \subseteq \{v_{c}, a\}$. If $v_{c}$ is a vertex of degree 2 in $H_{c}^{*}$, then let $N(v_{c}) = \{v_{0}, y\}$. Then $H_{c}^{*}$ has a nontrivial 2-edge-connected subgraph $H_{c}^{*}$ with at most two vertices of degree 2 and with $\{v_{0}, a\} \subseteq V(H_{c}^{*})$. By Corollary 2.1, $H_{c}^{*}$ and $H_{2}$ are collapsible. Therefore, $(G/\pi)/(H_{c}^{*} \cup H_{2}) = K_{3}$, and so $G/\pi \in \mathcal{CL}$. By Theorem D, $G \in \mathcal{CL}$, a contradiction. This proved the case $\kappa'(G/\pi) = 2$.

**Case 3.** $\kappa'(G/\pi) = 1$. Since $\delta(G/\pi) \geq 3$, by Lemma 2.2, $G$ is not reduced, a contradiction.

This completes the proof of Claim 3, and so Theorem 4.1(c)/ii) holds.

**Claim 4.** $D_{2}'(G)$ is an independent set.

By way of contradiction, suppose that $G$ has an edge $e = ab$ with
Lemma 5.1. Let $H$ be a spanning trail $D$ in $G$ by splitting each vertex $v$. Let $P$, $P_{14}$, $P = P_{14} + e$ has girth at most 4, contrary to Claim 3.

If $G = P$, then $G = P + e \neq P$. By Theorem B, $G \in \mathcal{CL}$, a contradiction. Claim 4 and Theorem 4.1(c)(iii) are proved.

Claim 5. $\Delta(G) \leq \lfloor \frac{|V(G)| - 1}{3} \rfloor$.

Let $\Delta(G) = t$. Let $v$ be a vertex with degree $d(v) = t$. Let $N(v) = \{x_1, x_2, x_3, \ldots, x_t\}$. Since $G$ has no 3- and 4-cycles, $(N(x_i) - v) \cap (N(x_j) - v) = \emptyset$. Since $\delta(G) \geq 3$, $|N(x_i)| = d(x_i) \geq 3$ and so

$$|V(G)| \geq 1 + t + \sum_{i=1}^{t} (|N(x_i)| - 1) \geq 1 + t + 2t = 1 + 3t.$$  

Since $\Delta(G) = t$ is an integer, $\Delta(G) \leq \lfloor \frac{|V(G)| - 1}{3} \rfloor$. Claim 5 is proved. □

5. Proof of Theorem 1.1

Lemma 5.1. Let $G$ be a 2-connected simple graph with $V(G) = D_3(G) \cup D_4(G)$. Let $D_4(G) = \{v_1, \ldots, v_4\}$. Let $G_1$ be a graph obtained from $G$ by splitting each vertex $v_i$ in $D_4(G)$ into two vertices $v_i^1$ and $v_i^2$ joint by an edge $e_i$ (see Figure 5.1) such that $G_1$ is a 3-regular graph with $V(G_1) = (V(G) - D_4(G)) \cup \{v_i^1, v_i^2\}$ and $E(G_1) = E(G) \cup \{e_1, e_2, \ldots, e_s\}$. Then

(a) $G_1$ is 2-connected with order $|V(G)| + |D_4(G)|$ and has the girth greater or equal to the girth of $G$;
(b) if $G_1$ is hamiltonian, then $G$ is supereulerian.

![Figure 5.1](attachment:figure51.png)

**Proof.** Lemma 5.1(a) follows from the definitions.

For (b), suppose that $G_1$ is hamiltonian. Let $H_0$ be a hamiltonian cycle in $G_1$. Let $E_0 = E(H_0) \cap \{e_1, e_2, \ldots, e_s\}$, then $G = G_1/\{e_1, e_2, \ldots, e_s\}$ has a spanning trail $H = H_0/E_0$. Lemma 5.1(b) is proved. □

**Proof of Theorem 1.1(a).** Let $G'$ be the reduction of $G$. By way of contradiction, suppose that $G$ is a smallest counterexample. Then $G = G'$,

$$G \notin \mathcal{CL} \text{ and } G \neq P.$$  

(7)
Therefore, by Theorem 4.1(c), $G$ is a reduced, 2-connected, 3-edge-connected and essentially 4-edge-connected graph with girth at least 5. By Theorem B, we only need to consider graph $G$ with $12 \leq |V(G)| \leq 13$.

**Case 1.** $|V(G)| = 12$. By Theorem 4.1(c)(iv), $3 \leq \Delta(G) \leq \lfloor \frac{|V(G)| - 1}{2} \rfloor$. Then $\Delta(G) = 3$. Thus, $G$ is a cubic 2-edge-connected Non-Hamiltonian graph of order 12 with girth at least 5, contrary to Theorem F(a).

**Case 2.** $|V(G)| = 13$. Since $\delta(G) \geq 3$, $\Delta(G) = 4$. Thus, $V(G) = D_2(G) \cup D_4(G)$ and $|D_4(G)|$ must be an odd number. If $|D_4(G)| \geq 5$, then

$$|E(G)| = \frac{4|D_2(G)| + 3|D_4(G)|}{2} \geq \frac{5 \times 4 + 8 \times 3}{2} = 22$$

and so $F(G) = 2|V(G)| - |E(G)| - 2 \leq 26 - 24 = 2$. By Theorem C(c) and $G \neq K_1, G \in \{K_2, K_{2,1}\}$, contrary to $\kappa'(G) \geq 3$. Thus, $|D_4(G)| = 1$ or 3.

Let $G_1$ be the graph obtained from $G$ by splitting the vertices in $D_4(G)$ as defined in Lemma 5.1. $G_1$ is a cubic 2-connected graph of order 14 or 16. Since $G$ has girth at least 5, $G_1$ has girth at least 5. By Theorem F(a), there is no 2-edge-connected cubic Non-Hamiltonian graph of order 14 or 16 with girth greater than 4, $G_1$ must be Hamiltonian. By Lemma 5.1, $G$ is supereulerian, contrary to (7).

This completes the proof of Theorem 1.1(a). $\Box$

Let $T_3$ be a path of length 3.

**Corollary 5.2.** Let $G$ be a connected simple graph with $|V(G)| \leq 13$ and $\delta(G) \geq 3$. Then $G' \in \{K_1, K_2, K_{1,2}, K_{1,3}, T_3, P\}$.

**Proof.** By Theorem 1.1(a), if $\kappa'(G) \geq 3$, then Corollary 5.2 follows. Thus we may assume that $\kappa'(G) \leq 2$. Let $X \subseteq E(G)$ be an edge cut of $G$ with $|X| \leq 2$. Let $G_1$ and $G_2$ be the two components of $G - X$ with $|V(G_1)| \leq |V(G_2)|$. Since $\delta(G) \geq 3$, it follows that $\delta(G_i) \geq 2$ ($i = 1, 2$) and

$$4 \leq |V(G_1)| \leq 6 \text{ and } |V(G_2)| = 13 - |V(G_1)|. \quad (8)$$

**Case 1.** $\kappa'(G) = 1$. Then $|D_2(G_i)| \leq 1$ ($i = 1, 2$). If $|V(G_1)| = 6$, then by (8) $|V(G_2)| \leq 7$. By Corollary 2.1(a) and (b), $G_1 \in \mathcal{CL}$ and the reduction of $G_2$ is in $\{K_1, K_2\}$. Hence the reduction of $G$ is in $\{K_1, K_2, K_{1,2}\}$.

If $4 \leq |V(G_1)| \leq 5$, then by (8), $8 \leq |V(G_2)| \leq 9$. By Corollary 2.1(a), $G_1 \in \mathcal{CL}$. By Corollary 2.1(d) the reduction of $G_2$ is in $\{K_1, K_2, K_{1,2}\}$ and so the reduction of $G$ is in $\{K_1, K_2, K_{1,2}, K_{1,3}, T_3\}$.

**Case 2.** $\kappa'(G) = 2$. Then $|D_2(G_i)| \leq 2$ and $\kappa'(G_i) \geq 2$ ($i = 1, 2$). Since $|V(G_1)| \leq 6$ with $|D_2(G_1)| \leq 2$, by Corollary 2.1(a), $G_1 \in \mathcal{CL}$. Since $|V(G_2)| \leq 9$ with $|D_2(G_2)| \leq 2$, by Corollary 2.1(c), $G_2 \in \{K_1, K_{2,3}\}$.

If $G'_2 = K_1$, then $G_2 \in \mathcal{CL}$, $G/(G_1 \cup G_2) = C_2 \in \mathcal{CL}$. Thus, $G' = K_1$. 

If $G' = K_{2,3}$, then let $D_2(K_{2,3}) = \{u_1, u_2, u_3\}$. If $H(u_i)$ is a non-trivial preimage of $u_i$, then since $\delta(G) \geq 3$, $|V(H(u_i))| \geq 4$. Since $|V(G_2)| \leq 9$ and $G'_2 = K_{2,3}$, $G_2$ has at most one nontrivial collapsible subgraph with at least 4 vertices. Thus, at least two vertices of degree 2 in $G'_2 = K_{2,3}$ are trivial contractions. Let $u_1$ and $u_2$ be the two trivial contractions of $G'_2$. Since $|X| = 2$ and $\delta(G) \geq 3$, $u_i$ must be incident with an edge in $X$ ($i = 1, 2$), and $u_3$ has a non-trivial preimage $H(u_3)$. Therefore, $G/(G_1 \cup H(u_3)) = K_{3,3} - e$.

By Corollary 2.1(a), $K_{3,3} - e$ is collapsible and so $G' = K_1$. □

**Remark.** Theorem 1.1(a) and Corollary 5.2 were first proved in [17] (without using the computer search results [23]). The proof outlined in [17] was long and complicated which involved checking on many cases and was never submitted for publication in journals. However, the result has been used by several authors [8, 9, 12]. Using that result (i.e., Theorem 1.1(a)), Catlin and Lai obtained the following:

**Theorem I** (Catlin and Lai [12]). If $G$ is a 3-edge-connected graph with at most 10 edge-cuts of size 3, then either $G \in \mathcal{S}$ or $G' = P$.

We will make use of Theorem I in the proof of Theorem 1.1(b) and (c).

**Proof of Theorem 1.1(b) and (c).** By Theorem 1.1(a), we only need to consider graphs $G$ with $14 \leq |V(G)| \leq 15$. By way of contradiction, suppose that $G$ is a counterexample with $|E(G)|$ minimized. Then $G$ is reduced. By Theorem 4.1(c), $G$ is a 2-connected and essentially 4-edge-connected reduced graph with girth at least 5, $\Delta(G) \leq 4$ and $D_4(G)$ is an independent set. Thus, $V(G) = D_3(G) \cup D_4(G)$.

**Case 1.** $|V(G)| = 14$ (Theorem 1.1(b)). Then $|D_4(G)|$ must be even.

If $|D_4(G)| = 0$, then by Theorem F, $G \in \{P, P_{14}\}$, a contradiction.

If $|D_4(G)| \geq 4$, then $|D_3(G)| \leq 10$. Since $G$ is essentially 4-edge-connected, $G$ is 3-edge-connected with at most 10 edge-cuts of size 3. By Theorem I, either $G \in \mathcal{S}$ or $G' = P$, a contradiction. Thus, $|D_4(G)| = 2$.

Let $G_1$ be the graph obtained from $G$ by splitting the two vertices in $D_4(G)$ as stated in Lemma 5.1. Then $G_1$ is a 2-connected cubic graph of order 16 with girth at least 5. By Theorem F, since $G_1 \neq P$ with girth at least 5, $G_1$ is hamiltonian. By Lemma 5.1, $G \in \mathcal{S}$, a contradiction.

**Case 2.** $|V(G)| = 15$ (Theorem 1.1(c)). Then $|D_4(G)|$ must be odd. If $|D_4(G)| = 1$, let $G_1$ be the graph obtained from $G$ by splitting the vertex in $D_4(G)$ as defined in Lemma 5.1. Then $G_1$ is a 2-edge-connected cubic graph of order 16 with girth at least 5. By Theorem F, all the cubic 2-edge-connected Non-Hamiltonian graphs of order 16 have girth at most 4. Thus, $G_1$ is a hamiltonian. By Lemma 5.1, $G \in \mathcal{S}$, a contradiction.

If $|D_4(G)| \geq 5$, then $|D_3(G)| = |V(G)| - |D_4(G)| \leq 10$. Since $G$ is essentially 4-edge-connected, $G$ is 3-edge-connected with at most 10 edge-
cuts of size 3. By Theorem 1, either \( G \in \mathcal{SL} \) or \( G' = P \), a contradiction. Thus, \( |D_4(G)| = 3 \) and so \( G \) is the graph defined in Theorem 1.1(c). \( \square \)

We conclude this paper with a conjecture that is a refinement of Conjecture 1:

**Conjecture 3.** Any 3-edge-connected simple graph of order at most 17 is either supereulerian or its reduction is in \{\( P, P_{14}, P_{16} \)\}.

**References**


REFERENCES


