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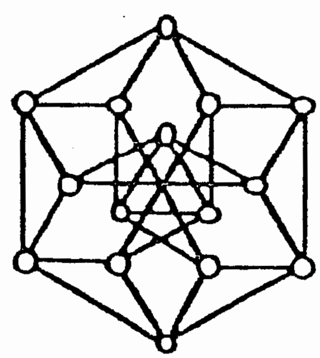

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Even subgraphs of a graph

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Abstract

In [Discrete Math. 101 (1992) 33 - 37], Fleischner proved that if G is a 2-edge-connected graph, then G has an even subgraph H with $\delta(H) \geq 2$ such that H contains all vertices of G with degree at least 3. In [J. Combinatorial Theory, Ser. B 35 (1983) 297 - 308], Bermond Jackson and Jaeger showed that every 2-edge-connected graph G has an even subgraph H with $|E(H)| \geq \frac{2}{3}|E(G)|$. In this note, we shall show that if G is a 2-edge-connected graph, then each of the following holds:

(i) G has an even subgraph H such that H contains all vertices of degree at least 3 in G and such that H contains a given pair of adjacent edges in G.

(ii) G has an even subgraph H such that H contains all vertices of degree at least 3 in G and such that $|E(H)| \geq \frac{2}{3}|E(G)|$.

raphs in this note are finite and undirected, and may have mul- edges and loops. For a graph G, we denote $O(G)$ the set of ces of odd degree in G. A graph G is even if $O(G) = \emptyset$. Let v be an edge in G. The contaction G/e is the graph obtained G by identifying the two ends of e and by deleting the resulting For each integer $i \geq 1$, denote

$$D_i(G) = \{v \in V(G) : d_G(v) = i\} \text{ and } D_i^*(G) = \cup_{j \geq i} D_j(G).$$

Jsing the Splitting Lemma (Lemma III.26 of [5], see also [6]) and Petersen 1-factor theorem, Fleischner in [4] proved the following theorem.

Theorem 1 (Fleischner, [4]) Let G be a nontrivial graph without cut edges. Then G has an even subgraph H such that $\delta(H) \geq 2$ and such that $V(G) - D_2(G) \subseteq V(H)$.

In [1], Bermond, Jackson and Jaeger proved the following:

Theorem 2 (Bermond, Jackson, and Jaeger, [1]) Every 2-edge-connected graph G has an even subgraph H with $|E(H)| \geq \frac{2}{3}|E(G)|$.

The main purpose of this note is to present some extensions of these two theorems by showing the following Theorem 3. Our method is a modification of the arguments in both [1] and [4].

Theorem 3 Let G be a 2-edge-connected graph. Then each of the following holds:

(i) G has an even subgraph H such that H contains all vertices of degree at lease 3 in G and such that H contains a given pair of adjacent edges in G.

(ii) G has an even subgraph H such that H contains all vertices of degree at lease 3 in G and such that $|E(H)| \geq \frac{2}{3}|E(G)|$.

The following Theorem 4 is needed. The proof for Theorem 3 follows from Lemmas 5 and 6 below.

Theorem 4 (Edmonds, [3]) Let G be a 2-edge-connected 3-regular graph. Then there is an integer $k \geq 1$ and a family of perfect matchings $(M_1, \cdots, M_{3k})$ such that each edge $e \in E(G)$ is in exactly k of the M_i's.

Let $v \in V(G)$. Define

$$E_G(v) = \{e \in E(G) : \ e \text{ is incident with } v \text{ in } G\}.$$

Lemma 5 Let G be a 2-edge-connected graph. For any $u \in V(G)$ and for any two edges $e_1, e_2 \in E_G(u)$, G has an even subgraph H satisfying each of the following properties:

(i) $\delta(H) \geq 2$,

(ii) $D_3^*(G) \subseteq V(H)$, and

(iii) $\{e_1, e_2\} \subseteq E(H)$.

Proof We argue by contradiction. Let G be a counterexample such that

$$\sum_{v \in D_4^*(G)} d_G(v) \text{ is minimized,} \tag{1}$$

and subject to (1),

$$|E(G)| \text{ is minimized.} \tag{2}$$

We have the following observations.

Claim 1. $\Delta(G) \leq 3$ and so $d_G(u) \leq 3$.

Suppose that $u \in D_4^*(G)$. Let $N_G(u) = \{u_1, \cdots, u_m\}$ where $e_i = uu_i$, $1 \leq i \leq 2$. Let G' be the graph obtained from G by splitting u into two vertices u' and u'' such that u' is exactly adjacent to u_1, u_2 and u'', and such that u'' is exactly adjacent to $u', u_3, \cdots u_m$. Note that if G' has a cut edge, then since G is 2-edge-connected, the cut edge in G' must be the new edge $u'u''$.

Case A1: $u'u''$ is a cut edge of G'.

Let G_1' and G_2' be the two components of $G' - u'u''$ such that $\{e_1, e_2\} \subseteq E(G_1')$. Since G is 2-edge-connected, G_1' and G_2' are also 2-edge-connected. By (1) and (2), G_1' has an even subgraph H_1 with $\delta(H_1) \geq 2$ and $D_3^*(G_1') \subseteq V(H_1)$, and $\{e_1, e_2\} \subseteq E(H_1)$. Similary, G_2' also contains an even subgraph H_2 such that $\delta(H_2) \geq 2$ and $D_3^*(G_2') \subseteq V(H_2)$. Therefore $H = G[E(H_1) \bigcup E(H_2)]$ is an even subgraph in G satisfying Lemma 5, contrary to the assumption that G is a counterexample.

Case A2: G' is 2-edge-connected.

By (1), G' has an even subgraph H' with $\delta(H') \geq 2$ and with $D_3^*(G') \subseteq V(H')$ such that $\{e_1, e_2\} \subseteq E(H')$.

Let $H = H'$ if $u'u'' \notin E(H')$ and $H = H'/\{u'u''\}$ if $u'u'' \in E(H')$. Then H will be the desired even subgraph in G, contrary to the assumption that G is a counterexample. This proves Claim 1.

Since G is 2-edge-connected, by Claim 1, we have $2 \leq \Delta(G) \leq 3$. If G is 2-regular, then the theorem holds tivially. If G is a 3-regular, then let e_3 be the only edge in $E_G(u) - \{e_1, e_2\}$. Since G is 2-edge-connected 3-regular graph, by Theorem 4, there is a perfect matching M of G such that $e_3 \in M$. It follows the $H = G - M$ is the desired even subgraph. A contradiction again.

Next we only need to consider that case that $\Delta(G) = 3$ and $D_2(G) \neq \emptyset$. Suppose that G has a vertex $w \in D_2(G)$.

Assume first that $w \neq u$ and that $E_G(w) = \{e', e''\}$. We may assume that $e'' \notin \{e_1, e_2\}$, since $w \neq u$. Then by (2), G/e'' has an even subgraph H'' with $\delta(H'') \geq 2$ and with $D_3^*(G/e'') \subseteq V(H'')$ such that $\{e_1, e_2\} \subseteq E(H'')$.

Let $H = G[E(H'')]$ if $e' \notin E(H'')$ and $H = G[E(H'') \cup \{e''\}]$ if $e' \in E(H')$. Then since $w \in D_2(G)$, H will be the desired even subgraph in G, contrary to the assumption that G is a counterexample.

Assume then $w = u \in D_2(G)$. If G is spanned by an edge e_1, then the theorem holds trivially. Assume that is not the case, and so there is an edge $e \in E(G) - E_G(u)$ such that e and e_1 are adjacent in G. By (2), G/e_1 has an even subgraph H_1 with $\delta(H_1) \geq 2$ and with $D_3^*(G/e_1) \subseteq V(H_1)$ such that $\{e, e_2\} \subseteq E(H_1)$. Thus by $u \in D_2(G)$, $G[E(H_1) \cup \{e_1\}]$ is a desired even subgraph, contrary to the assumption that G is a counterexample. This proves Lemma 5. □

A graph G is a weighted graph if G is associated with a non-negative integer valued function $w : E(G) \longrightarrow Z^+ \cup \{0\}$, ($w$ is called the weight function). If $X \subseteq E(G)$, then $w(X) = \sum_{e \in X} w(e)$. If H is a subgraph, then $w(H) = w(E(H))$.

<u>Lemma 6</u> Let G be a weighted graph with $\kappa'(G) \geq 2$ and with weight function w. Then G has an even subgraph H with $\delta(H) \geq 2$ and with $D_3^*(G) \subseteq V(H)$ such that $w(H) \geq \frac{2}{3} w(G)$.

<u>Proof</u> As in the proof of Lemma 5, we argue by contradiction and assume that G is a counterexample such that

$$\sum_{v \in D_4^*(G)} d_G(v) \text{ is minimized,} \tag{3}$$

and subject to (3),

$$|E(G)| \text{ is minimized.} \tag{4}$$

If $D_2(G) \neq \emptyset$, then let $v \in D_2(G)$ and let $E_G(v) = \{e_1, e_2\}$. Let G' denote the weighted graph obtained from $G - v$ by adding a new edge e joining the two neighbors of v in G, and by assigning the weight $w(e) = w(e_1) + w(e_2)$. By (4), G' has an even subgraph H' with

$$D_3^*(G') \subseteq V(H') \text{ and } w(H') \geq \frac{2}{3} w(G').$$

Note that $D_3^*(G') = D_3^*(G)$ and $w(G') = w(G)$. It follows that

$$H = \begin{cases} G[E(H')] & \text{if } e \notin E(H') \\ G[E(H' - e) \cup \{e_1, e_2\}] & \text{otherwise} \end{cases}$$

is the desired even subgraph. Hence we may assume that $\delta(G) \geq 3$.

Suppose that $u \in D_4^*(G)$. Let $N_G(u) = \{u_1, \cdots, u_m\}$ with $m \geq 4$. Let $e_i = uu_i$, $1 \leq i \leq 2$. Let G'' be the graph obtained from G by splitting u into two vertices u' and u'' such that u' is exactly adjacent to u_1, u_2 and u'', and such that u'' is exactly adjacent to $u', u_3, \cdots u_m$. Note that G'' may have $u'u''$ as an only cut edge since G is 2-edge-connnected. If this is the case, then interchange u_2 and u_3 can assume that the new graph G'' is 2-edge-connected. Let e denote the new edge joining u' and u''. Then one can view $E(G'') = E(G) \cup \{e\}$. Extend the domain of w by defining $w(e) = 0$. Then G'' with the extended w is a weighted graph. By (3), G'' has an even subgraph H'' such that

$$D_3^*(G'') \subseteq V(H'') \text{ and } w(H'') \geq \frac{2}{3} w(G'').$$

Note that $D_3^*(G) - \{u\} \subseteq D_3^*(G'')$ and $w(G) = w(G'')$. It follows that

$$H = \begin{cases} G[E(H')] & \text{if } e \notin E(H') \\ G[E(H/e)] & \text{otherwise} \end{cases}$$

is the desired even subgraph. Hence we may assume that $\delta(G) = 3$, and so G is 3-regular.

When G is 3-regular, Lemma 6 follows from Theorem 4. In fact, by Theorem 4, for some integer $k \geq 1$, G has a family of perfect matchings $(M_1, \cdots, M_{3k})$ such that each edge $e \in E(G)$ is in exactly k of the M_i's.

Assume that $w(M_1) \leq w(M_2) \leq \cdots \leq w(M_{3k})$. Then $3kw(M_1) \leq \sum_{i=1}^{3k} w(M_i) = kw(E(G))$, and so $w(M_1) \leq \frac{1}{3} w(E(G))$. It follows that $H = G - M_1$ is an even subgraph with $\delta(H) \geq 2$, $D_3^*(G) \subseteq V(H)$ and $w(H) \geq \frac{2}{3} w(E(G))$. The proof of Lemma 6 is complete. □

<u>Proof of Theorem 3</u>: Theorem 3(i) follows from Lemma 5 and Theorem 3(ii) follows from Lemma 6 with $w(e) = 1$. □

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