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Supereulerian graphs and the Petersen graph, II

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Abstract

In this note, we verify two conjectures of Catlin in [J. Graph Theory 13 (1989) 465 - 483] for graphs with at most 11 vertices. These are used to prove the following theorem which improves prior results in [10] and [13]:

Let G be a 3-edge-connected simple graph with order n. If n is large and if for every edge $uv \in E(G)$, $d(u) + d(v) \geq \frac{n}{6} - 2$, then either G has a spanning eulerian subgraph or G can be contracted to the Petersen graph.

1. Introduction. We follow the notation of Bondy and Murty [3], except that graphs have no loops. The graph of order 2 and size 2 is called a 2-cycle and denoted C_2, and K_1 is regarded as having infinite edge-connectivity. For $X \subseteq E(G)$, the *contraction* G/X is the graph obtained from G by identifying the two ends of each edge $e \in X$ and by deleting the resulting loops. If H is a subgraph of G, then we write G/H for $G/E(H)$. If H is connected, then v_H denotes the vertex in G/H to which H is contracted. We say that v_H is *nontrivial* if $E(H) \neq \emptyset$. For an integer $i \geq 1$, define

$$D_i(G) = \{v \in V(G) : d(v) = i\}.$$

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For a graph G, let $O(G)$ denoted the set of vertices of odd degree in G. A graph G is *eulerian* if it is connected with $O(G) = \emptyset$. The following was conjectured in [1] and was recently proved by Veldman [13].

Theorem 1.1 (Veldman [13]) Let G be a 2-edge-conenected simple graph with n vertices. If n is large and if for every edge $uv \in E(G)$,

$$d(u) + d(v) \geq \frac{2n}{5} - 2, \tag{1}$$

then either G has an eulerian subgraph that contains at least one end of every edge of G, or G can be contracted to $K_{2,3}$ such that the preimage of each of the vertices of degree 2 in this $K_{2,3}$ is nontrivial. □

For 3-edge-connected graphs, the lower bound in Theorem 1.1 can be improved with a stronger conclusion.

Theorem 1.2 (Chen and Lai [10], and Veldman [13]) Let G be a 3-edge-connected simple graph with n vertices. If n is large and if for every edge $uv \in E(G)$,

$$d(u) + d(v) \geq \frac{n}{5} - 2, \tag{2}$$

then either G has a spanning eulerian subgraph, or G can be contracted to the Petersen graph such that the preimage of each vertex of this Petersen graph is nontrivial. □

In this note, we shall further improve the lower bound in (2) of Theorem 1.2.

Theorem 1.3 Let G be a 3-edge-connected simple graph with n vertices. If n is large and if for every edge $uv \in E(G)$,

$$d(u) + d(v) \geq \frac{n}{6} - 2,$$

then either G has a spanning eulerian subgraph, or G has the Petersen graph as its reduction (we define *reduction* in section 2).

Theorem 1.3 is a special case of Theorem 3.1 in Section 3. In Section 2, we shall provide some mechanisms needed for the proof, and in Section 3, we present the proof of the main result.

2. Collapsible graphs and reduced graphs. A graph G is *supereulerian* if it has a spanning eulerian subgraph. G is *collapsible* if for every set $R \subseteq V(G)$ with $|R|$ even, there is a spanning connected subgraph H_R of G, such that $O(H_R) = R$. Thus K_1 is both supereulerian and collapsible.

Denote the family of supereulerian graphs by $\mathcal{SL}$, and denote the family of collapsible graphs by $\mathcal{CL}$. Let G be a collapsible graph and let $R = \emptyset$. Then by definition G has a spanning connected subgraph H with $O(H) = \emptyset$, and so G is supereulerian. Therefore, we have

$$\mathcal{CL} \subseteq \mathcal{SL}. \tag{3}$$

Examples of graphs in $\mathcal{CL}$ include the cycles C_2, C_3, but not C_t if $t \geq 4$.

In [5], Catlin showed that every graph G has a unique collection of pairwise disjoint maximal collapsible subgraphs $H_1, H_2, \cdots, H_c$. The contraction of G obtained from G by contracting each H_i into a single vertex, $(1 \leq i \leq c)$, is called the *reduction* of G. A graph is *reduced* if it is the reduction of some other graph.

Let $F(G)$ denote the minimum number of extra edges that must be added to G so that the resulting graph has 2 edge-disjoint spanning trees. Catlin showed in [6, Theorem 7] that if G is reduced, then

$$F(G) = 2|V(G)| - |E(G)| - 2. \tag{4}$$

Theorem 2.1 (Catlin [5]) Let G be a graph.

(a) (Theorem 5 of [5]) G is reduced if and only if G has no nontrivial collapsible subgraph. Thus, every subgraph of a reduced graph is also reduced.

(b) (Corollary 1 and Theorem 2 of [5]) If G has a spanning tree T such that every edge of T is in a collapsible subgraph of G , or if G has two edge-disjoint spanning trees, then G is collapsible.

(c) (Theorem 3 of [5]) Let H be a collapsible subgraph of G. Then G is collapsible if and only if G/H is collapsible, and G is supereulerian if and only if G/H is supereulerian.

(d) (Theorem 8 of [5]) If $G \notin \{K_1, K_2\}$ is reduced, then G is simple and K_3-free with $\delta(G) \leq 3$, and $F(G) \geq 2$.

Let G be a graph containing a 4-cycle $C = uvzwu$. Following Catlin [6], we define $G/\pi(C)$ to be the graph obtained from $G - E(C)$ by identifying u and z to form a vertex x, by identifying v and w to form a vertex y, and by adding an edge $e_\pi = xy$.

Theorem 2.2 (Catlin, Theorem 10 of [6]) Let G containing a 4-cycle C, be given and let $G/\pi(C)$ be defined as above. If $G/\pi(C) \in \mathcal{CL}$, then $G \in \mathcal{CL}$. □

Lemma 2.3 The graphs $L_1, L_2, L_3, L_4, L_5, L_6$ and L_7 defined in Figure 1 are all collapsible.

Proof: By Theorem 11 of [6], $L_1 \in \mathcal{CL}$. The graph L_2 can be obtained from L_1 by contracting an edge, and so it is routine to verify that $L_2 \in \mathcal{CL}$. Denote $C = uvzwu$ in L_i, for all $i \geq 3$. Since $L_3/\pi(C) = L_2 \in \mathcal{CL}$ and $L_4/\pi(C) = L_2 \in \mathcal{CL}$, we have $L_3, L_4 \in \mathcal{CL}$, by Theorem 2.2. Denote $C' = u'v'z'w'u'$ in L_5. Then $(L_5/\pi(C))/\pi(C')$ becomes a 3-cycle, after its parallel edges are contracted, and a 3-cycle is collapsible. Hence, by repeated application of Theorems 2.1 and 2.2, we conclude that $L_5 \in \mathcal{CL}$ also. Note that $L_6/\pi(C)$ has a unique 3-cycle C_3 and that every edge of $(L_6/\pi(C))/C_3$ lies in a 3-cycle, and so by Theorem 2.1(b), $(L_6/\pi(C))/C_3 \in \mathcal{CL}$. By (c) of Theorem 2.1, $L_6/\pi(C) \in \mathcal{CL}$, and so by Theorem 2.2, $L_6 \in \mathcal{CL}$. Note that $L_7/\pi(C)$ has a unique 2-cycle C_2 and a unique 3-cycle C_3, and that $((L_7/\pi(C))/C_2)/C_3 = K_3 \in \mathcal{CL}$ by Theorem 2.1(c). Therefore by Theorem 2.1(b), and by Theorem 2.2, $L_7 \in \mathcal{CL}$. □

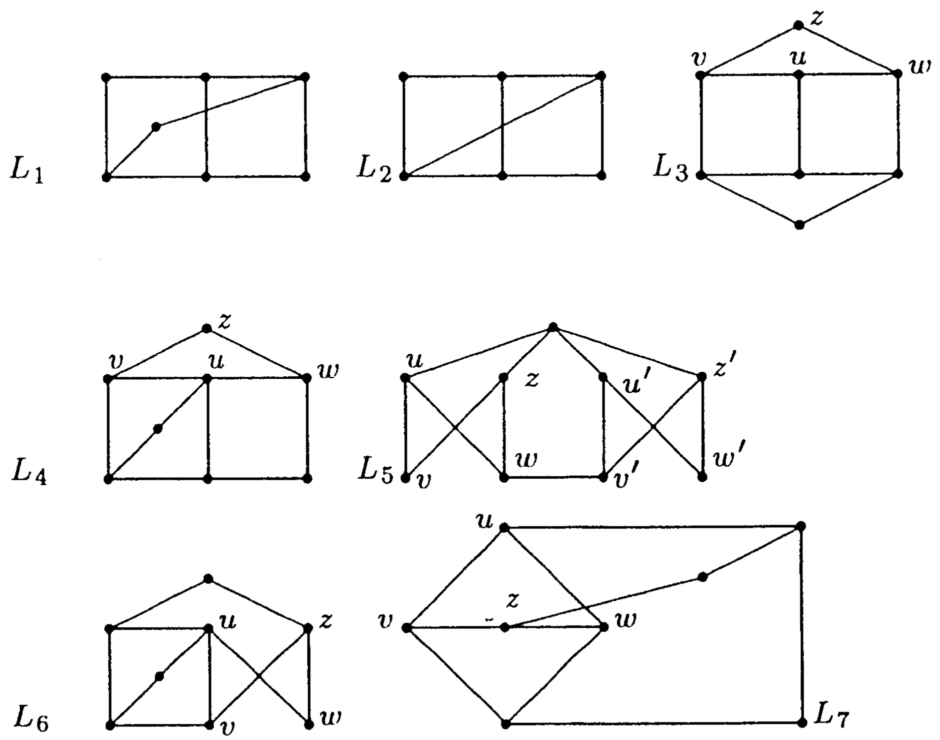

Figure 1: The graphs in Lemma 2.3

Definition of $\mathcal{F}$: The Petersen graph is denoted by P. Let s_1, s_2, s_3, m, l, t be natural numbers with $t \geq 2$ and $m, l \geq 1$. Let $M \cong K_{1,3}$ with center a and ends a_1, a_2, a_3. Define $K_{1,3}(s_1, s_2, s_3)$ to be the graph obtained from M by adding s_i vertices with neighbors $\{a_i, a_{i+1}\}$, where $i \equiv 1, 2, 3 \pmod 3$. Let $K_{2,t}(u, u')$ be a $K_{2,t}$ with u, u' being the nonadjacent vertices of degree t. Let $K'_{2,t}(u, u', u'')$ be the graph obtained from a $K_{2,t}(u, u')$ by adding a new vertex u'' that joins to u' only. Hence u'' has degree 1 and u has degree t in $K'_{2,t}(u, u', u'')$. Let $K''_{2,t}(u, u', u'')$ be the graph obtained from a $K_{2,t}(u,u')$ by adding a new vertex u'' that joins to a vertex of degree 2 of $K_{2,t}$. Hence u'' has degree 1 and both u and u' have degree t in $K''_{2,t}(u,u',u'')$. We shall use $K'_{2,t}$ and $K''_{2,t}$ for a $K'_{2,t}(u,u',u'')$ and a $K''_{2,t}(u,u',u'')$, respectively. Let $S(m,l)$ be the graph obtained from a $K_{2,m}(u,u')$ and a $K'_{2,l}(w,w',w'')$ by identifying u with w, and w'' with u'; let $J(m,l)$ denote the graph obtained from a $K_{2,m+1}$ and a $K'_{2,l}(w,w',w'')$ by identifying w, w'' with the two ends of an edge in $K_{2,m+1}$, respectively; let $J'(m,l)$ denote the graph obtained from a $K_{2,m+2}$ and a $K'_{2,l}(w,w',w'')$ by identifying w, w'' with two vertices of degree 2 in $K_{2,m+2}$, respectively. See Figure 2 for examples of these graphs. Let

$$\mathcal{F} = \{K_1,\ K_2,\ K_{2,t},\ K'_{2,t},\ K''_{2,t},\ K_{1,3}(s,s',s''),\ S(m,l),\ J(m,l),\ J'(m,l),\ P\},$$

where t, s, s', s'', m, l are nonnegative integers.

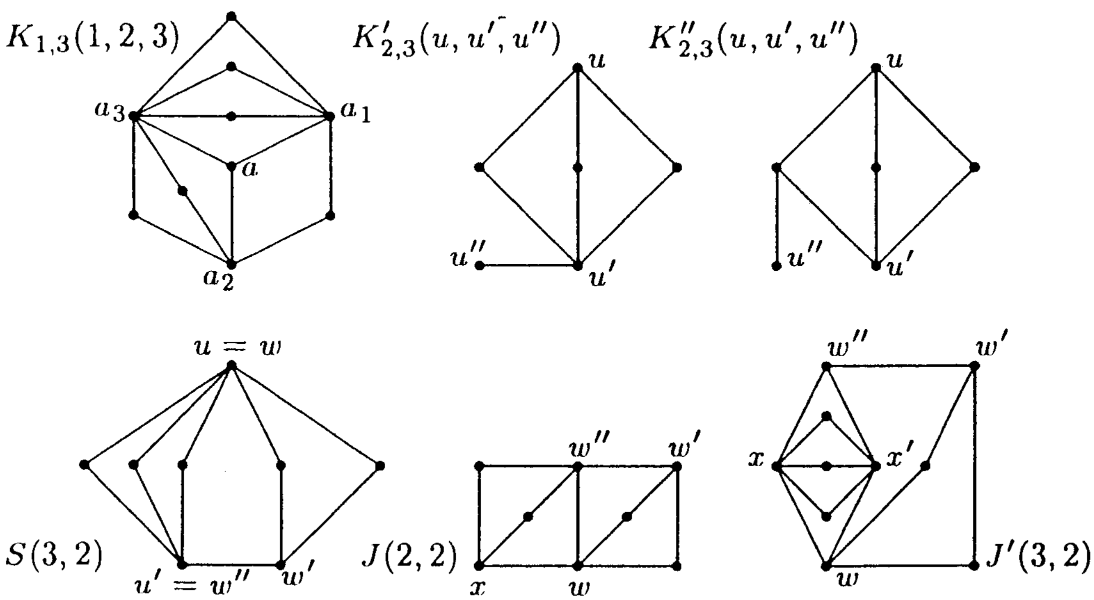

Figure 2: Some graphs in $\mathcal{F}$ with small parameters

<u>Theorem 2.4</u> If G is a connected reduced graph with $|V(G)| \leq 11$ and $F(G) \leq 3$, then $G \in \mathcal{F}$.

Catlin conjectured (Conjecture 3 of [7]) that a 2-edge-connected nontrivial reduced graph G with $F(G) = 2$ must be a $K_{2,t}$ for some $t \geq 2$, and (Conjecture 4 of [7]) that a 3-edge-connected nontrivial reduced graph G with $F(G) = 3$ must be the Petersen graph P. Theorem 2.4 indicates that both conjectures are valid for graphs with at most 11 vertices. We need the following to prove Theorem 2.4.

<u>Theorem 2.5</u> (Chen [8]) Let G be a reduced graph of order at most 11 with $\kappa'(G) \geq 3$, then $G \in \{K_1, P\}$. $\square$

Lemma 2.6 (Lemma 4 in Chapter 2 of [11]) Let $w \notin V(P)$ be a vertex and let H be a graph with $V(H) = V(P) \cup \{w\}$ and $E(P) \subseteq E(H)$. If w is adjacent to at least two distinct vertices of P, then $H \in \mathcal{CL}$. □

Proof of Theorem 2.4: In the proofs below, we shall use the notation in the definition of $\mathcal{F}$, which is illustrated in Figure 2. Note that trees with at most 3 edges are in $\mathcal{F}$. We also observe that $F(G) = 3$ for $G \in \mathcal{F} - \{K_1, K_2, K_{2,t}\}$; this can be checked with the help of Figure 2, and by observing that (4) is invariant with respected to vertex delection/addition if the vertex has degree 2. To obtain a contradiction, we assume that

$$G \text{ is a minimum counterexample to Theorem 2.4.} \tag{5}$$

By Theorem 2.5, we may assume $\kappa'(G) \le 2$. Also, by Theorem 2.1(d), $|V(G)| \ge 3$ and G is K_3-free.

Suppose first that $\kappa'(G) = 1$. Let e be a cut-edge with G_1 and G_2 being the two components of $G - e$. Thus by (4), and the hypothesis,

$$F(G_1) + F(G_2) = F(G) - 1 \le 2.$$

If $F(G_1) \le 1$ and $F(G_2) \le 1$, then by Theorem 2.1(d), both G_1 and G_2 are in $\{K_1, K_2\}$, and so G must be a tree with at most 3 edges, contrary to (5). Hence we may assume that $F(G_1) = 0$ and $F(G_2) = 2$. The minimality of G implies G_1, $G_2 \in \mathcal{F}$. $F(G_1) = 0$ and Theorem 2.1(d) imply $G_1 = K_1$. By (4), $K_{2,t}$ is the only member in $\mathcal{F}$ with F having value 2, and so we have $G_2 = K_{2,t}$, for some $t \ge 1$. It follows that $G = K'_{2,t} \in \mathcal{F}$, a contradiction.

Therefore from now on we assume that $\kappa'(G) = 2$. Let $\{e_1, e_2\}$ be an edge-cut with G_1 and G_2 being the two components of $G - \{e_1, e_2\}$. Then by (4), and the hypothesis,

$$F(G_1) + F(G_2) = F(G) \le 3.$$

Assume further that $F(G_1) = 1$ and $F(G_2) = 2$. By Theorem 2.1(d), $G_1 = K_2$. Therefore, e_1 and e_2 are independent edges (otherwise G would not be K_3-free). By the minimality of G, $G_2 = K_{2,t}$ for some $t \ge 1$. $G \ne K'_{2,2}$ since $\kappa'(K'_{2,2}) = 1$. It follows that either $G = S(1,1)$ (when $t = 1$), or $G \in \{J(1, t-1), S(1,t), J'(t-2, 1)\}$ (when $t \ge 2$), contrary to (5) in any case.

Hence we may assume that $F(G_1) = 0$ and $F(G_2) \in \{2, 3\}$ (Note that $F(G_1) = 0$, $F(G_2) = 1$ renders $G = K_3$ which is not reduced). By Theorem

2.1(d), $G_1 = K_1$. By the minimality of G, $G_2 \in \mathcal{F}$. By Lemma 2.6, $G_2 \neq P$. Let v denote the only vertex in $V(G_1)$ and v', v'' the two vertices adjacent to v in G. Since G is reduced, G has no 2-cycles and 3-cycles (by Theorem 2.1(d)), and so $v' \neq v''$ and

$$v'v'' \notin E(G). \tag{6}$$

Assume first that $G_2 = K_{2,t}$. If v' and v'' are the two vertices of degree t in G_2, then $G = K_{2,t+1}$ and so $G \in \mathcal{F}$, a contradiction. Therefore, by (6) we may assume that $t \geq 3$ and v' and v'' are in $D_2(G_2)$. However, G then has L_2 as a subgraph, contrary to the assumption that G is reduced.

The proof when G_2 is another member in $\mathcal{F}$ is similar. □

3. The main result and its proof. We shall prove a slightly more general result than Theorem 1.3. For an edge subset $X = \{x_iy_i : (1 \leq i \leq k)\} \subseteq E(G)$, define

$$\sum_G(X) = \sum_{i=1}^{k} d_G(x_i) + d_G(y_i).$$

Theorem 3.1 Let G be a simple graph with $n = |V(G)| > 306$ and with $\kappa'(G) \geq 3$. If for every matching M_6 of size 6 in G,

$$\sum_G(M_6) \geq n - 12, \tag{7}$$

then either $G \in \mathcal{SL}$ or G has the Petersen graph as its reduction.

The proof of Theorem 3.1 requires some prior results and some more lemmas. With ad hoc arguments similar to the proof of Theorem 2.5, Chen was able to make the following improvement of Theorem 2.5.

Theorem 3.2 (Chen [9]) Let G be a connected simple graph with $|V(G)| \leq 13$ and $\delta(G) \geq 3$. Then either G is a supereulerian graph with 12 vertices and with an odd cycle, or the reduction of G is in $\{K_1,\ K_2,\ K_{1,2},\ K_{1,3},\ P\}$. □

As in [3], $\alpha'(G)$ and $o(G)$ denote the maximum size of a matching in G and the number of odd components of G, respectively.

Theorem 3.3 (Berge [2] and Tutte [12]) Let G be a graph of n vertices. If

$$t = \max_{S \subset V(G)} \{o(G - S) - |S|\}, \tag{8}$$

then $\alpha'(G) = \dfrac{n-t}{2}$. □

Lemma 3.4 Let G be a bipartite graph with bipartition $\{X, Y\}$ such that $|X| = 6$ and $|Y| = 8$ and such that each vertex in Y has degree at least 3 in G. Then G is not reduced.

Proof: Assume to the contrary that G is reduced. Note that $|E(G)| \geq 3|Y| = 24$, and so by (4),

$$F(G) = 2|V(G)| - |E(G)| - 2 \leq 2. \tag{9}$$

Case 1: There exist $v_1, v_2, v_3 \in V(G)$ such that v_1 has degree at most 2 in G, v_2 has degree at most 2 in $G - v_1$, and v_3 has degree at most 2 in $G - \{v_1, v_2\}$.

Then by (4), $F(G - \{v_1, v_2, v_3\}) \leq 2$. Since G is reduced, by Theorem 2.4, we have $G - \{v_1, v_2, v_3\} \cong K_{2,9}$. Since $K_{2,9}$ is a connected subgraph of G, X or Y has at least 9 vertices, a contradiction.

Case 2: There is a subset $V' \subset V(G)$ with $1 \leq |V'| \leq 2$ and $\delta(G - V') \geq 3$. Since G is reduced, and by Theorem 3.2, $G - V'$ is either supereulerian with 12 vertices and with an odd cycle, or $G - V' = P$. Therefore $G - V'$ contains an odd cycle and so cannot be a subgraph of a bipartite graph G, a contradiction.

Case 3: $\delta(G) \geq 3$ and both Case 1 and Case 2 do not hold.
By (d) of Theorem 2.1, $\delta(G) = 3$ and so there is a vertex v_1 of degree 3 in G. Since Case 2 does not hold, there are vertices $v_2, v_3 \in V(G)$ so that v_2 has degree 2 in $G - \{v_1\}$ and v_3 has degree at most 2 in $G - \{v_1, v_2\}$. By (4), $F(G - \{v_1, v_2\}) \leq 3$. By Theorem 2.4, $G - \{v_1, v_2, v_3\} \in \mathcal{F} - \{K_1, K_2\}$. Let

$$\mathcal{F}'' = \{K_{2,9}, K'_{2,8}, K''_{2,8}, K_{1,3}(s, s', s''),\ (s+s'+s'' = 7), J(m, l),\ (m+l = 8)\}.$$

Since $S(m, l)$, $J'(m, l)$ and P have odd cycles, by Theorem 2.4,

$$G - \{v_1, v_2, v_3\} \in \mathcal{F}''.$$

Since each member of $\mathcal{F}''$ has at least 7 vertices of degree 2 and since there are at most 5 edges in G between $V(G) - \{v_1, v_2, v_3\}$ and v_1, v_2, v_3, G has at least one vertex of degree at most 2, contrary to the assumption of $\delta(G) \geq 3$. □

Lemma 3.5 Let G be a reduced graph with $n = |V(G)| \leq 14$ vertices and with $\delta(G) \geq 3$. Then

$$\alpha'(G) \geq \frac{n-1}{2}.$$

Proof: Define t by (8). It suffices to show $t = 1$. Assume to the contrary that $t \geq 2$. Let $S \subset V(G)$ be chosen such that $t = o(G - S) - |S|$.

Claim. G is connected and $S \neq \emptyset$.

It was proved in [8, Lemma 1] that a simple 2-edge-connected graph H of order at most 7 with $\delta(H) \geq 2$ and $|D_2(H)| \leq 2$ is collapsible. Since a reduced graph is a simple graph (Theorem 2.1(d)), and every subgraph of a reduced graph is also reduced (Theorem 2.1(a)), and since G is reduced with $|V(G)| \leq 14$ and $\delta(G) \geq 3$, G must be connected. It follows that $S \neq \emptyset$ since $t \geq 2$. The claim is proved.

For each odd integer i, let $\mathcal{R}_i$ be the collection of components of $G - S$ consisting of exactly i vertices, and let $r_i = |\mathcal{R}_i|$. Define

$$R_i = \cup_{H \in \mathcal{R}_i} V(H), \text{ and } G'' = G[R_1 \cup R_3 \cup S].$$

Then, by Theorem 2.1(a), G'' is a reduced graph with

$$n'' = |V(G'')| = |S| + r_1 + 3r_3. \tag{10}$$

For a component H in $G - S$, let ∂H denote the subset of $E(G)$ such that $e \in \partial H$ if and only if e is incident with at least one vertex in $V(H)$. Since $\delta(G) \geq 3$, we have

$$|\partial H| \geq 3, \text{ for any } H \in \mathcal{R}_1. \tag{11}$$

Since G is reduced, by (d) of Theorem 2.1, G does not have a K_3 as a subgraph, and so by $\delta(G) \geq 3$,

$$|\partial H| \geq 7, \text{ for any } H \in \mathcal{R}_3. \tag{12}$$

For any $H, H' \in \mathcal{R}_1 \cup \mathcal{R}_3$ with $H \neq H'$, since H and H' are distinct components of $G - S$, $\partial H \cap \partial H' = \emptyset$. Therefore by (11) and (12), we have

$$3r_1 + 7r_3 \leq |E(G'')|. \tag{13}$$

Note that since $t \geq 2$ and $S \neq \emptyset$, $o(G - S) = t + |S| \geq 3$. This, together with $|V(G)| \leq 14$, implies that r_1 and r_3 cannot be both zero. Thus, by (13), $G'' \notin \{K_1, K_2\}$. By Theorem 2.1(d), $F(G'') \geq 2$. This and (4) now yield

$$|E(G'')| \leq 2n'' - 4,$$

which, together with (10) and (13), gives

$$3r_1 + 7r_3 \leq |E(G'')| \leq 2|S| + 2r_1 + 6r_3 - 4,$$

and so

$$r_1 + r_3 \leq 2|S| - 4. \tag{14}$$

Let $q = o(G - S)$. Counting the vertices in G, we have

$$n \geq |S| + r_1 + 3r_3 + 5(q - r_1 - r_3), \tag{15}$$

and so by (8), by (15), by $n \leq 14$ and by (14),

$$\begin{aligned} t = q - |S| \quad &\leq \quad \frac{n - 6|S| + 4r_1 + 2r_3}{5} \\ &\leq \quad \frac{14 - 6|S| + 4(r_1 + r_3)}{5} \leq \frac{2|S| - 2}{5}. \end{aligned} \tag{16}$$

It follows by $t \geq 2$ that $|S| \geq 6$. By $t \geq 2$ and the inequality in (16), by (14), and by $r_1 \leq n - |S|$, we have

$$\begin{aligned} 10 \quad &\leq \quad n - 6|S| + 4r_1 + 2r_3 \\ &= \quad n - 6|S| + 2(r_1 + r_3) + 2r_1 \\ &\leq \quad n - 6|S| + 2(2|S| - 4) + 2(n - |S|) \\ &= \quad 3n - 4|S| - 8, \end{aligned} \tag{17}$$

and so $18 + 4|S| \leq 3n$ which together with $|S| \geq 6$ and $n \leq 14$ implies $n = 14$. Moreover, all inequalities used in establishing (16) and (17) become equations yielding $r_3 = 0$, $r_1 = 8$ and $|S| = 6$. Thus, $G'' = G[R_1 \cup S] = G$, and $H = G - E(G[S])$ is a spanning bipartite subgraph of G with bipartition R_1 and S satisfying the hypothesis of Lemma 3.4. Thus, H is not reduced and so by Theorem 2.1(a), G is not reduced either, contradicting the hypothesis and finally implying $t = 1$. □

Theorem 3.6 (Chen and Lai, Theorem 3 of [10]) Let G be a noncollapsible 3-edge-connected graph with n vertices, let G' be the reduction of G, and let $p \leq \alpha'(G')$ be a positive integer. If for every matching M_p of size p in G

$$\sum_G (M_p) \geq n - 2p,$$

and if $n \geq 3p(3p - 1)$, then we have $\kappa'(G') \geq 3$, $\alpha'(G') = p$ and $|V(G')| \leq 3p - 4$. □

Lemma 3.7 (Corollary 2 of [10]) If G is a nontrivial connected reduced graph with $\kappa'(G) \geq 3$, then $\alpha'(G) \geq \dfrac{|V(G)|+4}{3}$.

Lemma 3.8 If G is a reduced graph with $\kappa'(G) \geq 3$ and $\alpha'(G) \leq 5$, then $G \in \{K_1, P\}$. □

Proof: By Lemma 3.7 with $\alpha'(G) \leq 5$, we have $|V(G)| \leq 11$ and so Lemma 3.8 follows from Theorem 2.5. □

Proof of Theorem 3.1: Since every collapsible graph is supereulerian (by (3)), we may assume that $G \notin \mathcal{CL}$. Let G' denote the reduction of G. By the definition of contraction, we have $\kappa'(G') \geq \kappa'(G) \geq 3$, and so $\delta(G') \geq \kappa'(G') \geq 3$. Then $G' \neq K_1$, since $G \notin \mathcal{CL}$. If $\alpha'(G') \leq 5$, then by Lemma 3.8, $G' = P$. Hence we assume that $\alpha'(G') \geq 6$. Applying Theorem 3.6 with $p = 6$, we have $|V(G')| \leq 14$ and $\alpha'(G') = 6$. If $|V(G')| = 14$, then by Lemma 3.5, $\alpha'(G') = 7$ a contradiction. Hence $|V(G')| \leq 13$ and so by Theorem 3.2, either $G \in \mathcal{SL}$ or the reduction of G is the Petersen graph. □

One now can easily see that Theorem 1.3 follows from Theorem 3.1.

4. Open problem. We conclude this note with an open problem. Let G be a 3-edge-connected simple graph with $n = |V(G)|$. We conjecture that if n is large and if for every edge $uv \in E(G)$,

$$d(u) + d(v) > \frac{n}{9} - 2, \tag{18}$$

then either G has a spanning eulerian subgraph, or G can be contracted to the Petersen graph. (The Petersen graph does not have to be the reduction of G.)

This conjecture, if true, will be best possible in the following sense. Let B denote a Blanuša snark of order 18. (See a survey of Watkins and Wilson [14] for snarks). Obtain a graph $G(n)$ of order $n = 18m$ from B by replacing each vertex of B by a complete subgraph K_m. Then for every edge $uv \in E(G(n))$,

$$d(u) + d(v) \geq \frac{n}{9} - 2.$$

However, neither has G a spanning eulerian subgraph, nor is G contractible to the Petersen graph.

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