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Abstract

A graph G is supereulerian if it contains a spanning eulerian subgraph. Let n, m and p be natural numbers, $m, p \geq 2$. Let G be a 2-edge-connected simple graph on $n > p + 6$ vertices containing no K_{m+1}. We prove that if

$$|E(G)| \geq \binom{n-p+1-k}{2} + (m-1)\binom{k+1}{2} + 2p - 4, \qquad (1)$$

where $k = \left\lfloor \frac{n-p+1}{m} \right\rfloor$, then either G is supereulerian, or G can be contracted to a nonsupereulerian graph of order less than p, or equality holds in (1) and G can be contracted to $K_{2,p-2}$ (p is odd) by contracting a complete m-partite graph $T_{m,n-p+1}$ of order $n - p + 1$ in G. This is a generalization of the previous results in [3] and [5].

1. Introduction

We follow the notation of Bondy and Murty [1], except that graphs have no loops. For a graph G, the order of the maximum complete subgraph of G is called clique number of G and denoted by $cl(G)$. A graph is eulerian if it is connected and every vertex has even degree. A graph G is called supereulerian if it has a spanning eulerian subgraph H. A cycle C of G is called a hamiltonian cycle if $V(C) = V(G)$ and is called dominating cycle if $E(G - V(C)) = \emptyset$. A graph is hamiltonian if it contains a hamiltonian cycle. Obviously, hamiltonian graphs are special supereulerian graphs.

There is rich literature on the following extremal graph theory problems: for a given family $\mathcal{F}$ of graphs and for a natural number n, what is the maximum size of simple graphs of order n which are not in $\mathcal{F}$. For example, when $\mathcal{F} = \{$graphs with clique number at least $m\}$, this is Turán's Theorem. In this note, we consider the family

$$\mathcal{F} = \{\text{supereulerian graphs with clique number } m\}.$$

In fact, our results are related to Turán's Theorem.

Let G be a graph, and let H be a connected subgraph of G. The contraction G/H is the graph obtained from G by contracting all edges of H, and by deleting any resulting loops. Even when G is simple, G/H may not be.

Here are some prior results related to our subject.

Theorem A (Ore [8] and Bondy [2]). Let G be a simple graph on n vertices. If

$$|E(G)| \geq \binom{n-1}{2} + 2, \tag{2}$$

then exactly one of the following holds:

(a) G is hamiltonian;

(b) Equality holds in (2), and $G \in \{K_1 \vee (K_1 + K_{n-2}), K_2 + K_3^c\}$ (where K_3^c is the complement of K_3). $\square$

Theorem B (Veldman [10]). Let G be a 2-edge-connected simple graph of order n. If

$$|E(G)| \geq \binom{n-4}{2} + 11,$$

then G has a dominating cycle.$\square$

Theorem C (Cai [3]). Let G be 2-edge-connected simple graph on n vertices. If

$$|E(G)| \geq \binom{n-4}{2} + 6, \tag{3}$$

then exactly one of the following holds:

(a) G is supereulerian;

(b) $G = K_{2,5}$;

(c) Equality holds in (3), and either $G = Q_3 - v$ (the cube minus a vertex), or G contains a complete subgraph $H = K_{n-4}$ such that $G/H = K_{2,3}$. $\square$

Theorem D (Catlin and Chen [5]). Let G be a 3-edge-connected simple graph on n vertices. If

$$|E(G)| \geq \binom{n-9}{2} + 16,$$

then G is supereulerian. $\square$

In this paper, following closely the method of [5], we shall generalize Theorem C and Theorem D. In particular, we found that if a graph G is K_3-free or has small clique number then the lower bound of the inequalities in Theorem C and Theorem D can be improved.

2. **Notation and Turán's Theorem**

Let n and m be natural numbers, we define $t(m,n)$ as the following;

$$t(m,n) = \binom{n-k}{2} + (m-1)\binom{k+1}{2},$$

where $k = \lfloor \frac{n}{m} \rfloor$. It is easy to see that if $m = n$ or $m > n$ then $k = 1$ or $k = 0$, respectively, and so the right side of the equation above is equal to $\binom{n}{2}$. If $m = 2$ then

$$t(2,n) = \begin{cases} \frac{n^2}{4} & \text{if } n \text{ is even;} \\ \frac{n^2-1}{4} & \text{if } n \text{ is odd.} \end{cases}$$

Note that for $m > n$,

$$t(2,n) < t(3,n) < \cdots < t(n-1,n) < t(n,n) = t(m,n) = \binom{n}{2}. \tag{4}$$

One can see that $t(m,n)$ is related to the Turán numbers below.

For $m \leq n$, denote by $T_{m,n}$ the <u>complete m-partite graph</u> of order n with

$$\left\lfloor \frac{n}{m} \right\rfloor, \left\lfloor \frac{n+1}{m} \right\rfloor, \ldots, \left\lfloor \frac{n+m-1}{m} \right\rfloor$$

vertices in the various independent classes. Note that $T_{m,n}$ is the unique complete m-partite graph of order n whose independent classes are as equal as possible and $T_{n,n} = K_n$. Let $k = \lfloor \frac{n}{m} \rfloor$, it is known that the size of $T_{m,n}$ is

$$|E(T_{m,n})| = t(m,n) = \binom{n-k}{2} + (m-1)\binom{k+1}{2}.$$

<u>Theorem E</u> (Turán [9]). Let m and n be natural numbers, $m \geq 2$. Then every graph of order n and size greater than $|E(T_{m,n})|$ contains a K_{m+1}. Furthermore, $T_{m,n}$ is the only graph of order n and size $|E(T_{m,n})|$ that does not contain a K_{m+1}. $\square$

<u>Remark</u>. Let G be a graph of order n with maximum size that does not contain a K_{m+1}. If $m > n$ then $|E(G)| = \binom{n}{2}$. If $m \leq n$ then by Theorem E $|E(G)| \leq |E(T_{m,n})|$. Thus, if G is a graph containing no K_{m+1} then $|E(G)| \leq t(m,n)$. For convenience, we define

$$H_{m,n} = \begin{cases} T_{m,n} & \text{if } m < n; \\ K_n & \text{if } m \geq n. \end{cases}$$

3. **Catlin's Reduction Method**

The following concept was given by Catlin [4].

For a graph G, let $O(G)$ denoted the set of vertices of odd degree in G. A graph G is called collapsible if for every even set $X \subseteq V(G)$ there is a spanning connected subgraph H_X of G, such that $O(H_X) = X$. The trivial graph K_1 is both supereulerian and collapsible. The cycles C_2 and C_3 are collapsible, but C_t is not if $t \geq 4$. In fact, if G is collapsible then G contains a spanning (u, v)-trail for any $u, v \in V(G)$. In particular, a collapsible graph is supereulerian.

In [4], Catlin showed that every graph G has a unique collection of disjoint maximal collapsible subgraphs $H_1, H_2, \cdots, H_c$. Define G' to be the graph obtained from G by contracting each H_i into a single vertex, $(1 \leq i \leq c)$. Since $V(G) = V(H_1) \cup \cdots \cup V(H_c)$, the graph G' has order c. We call the graph G' the reduction of G. Any graph G has a unique reduction G' [4]. A graph G is reduced if $G = G'$.

We shall make use of the following theorems:

Theorem F (Catlin [4]) Let G be a graph. Let G' be the reduction of G.

(a) Let H be a collapsible subgraph of G. Then G is collapsible if and only if G/H is collapsible. In particular, G is collapsible if and only if $G' = K_1$.

(b) G is supereulerian if and only if G' is supereulerian.

(c) If G is a reduced graph of order n, then G is simple and K_3-free with $\delta(G) \leq 3$ and either $G \in \{K_1, K_2\}$, or
$$|E(G)| \leq 2n - 4.\Box$$

Theorem G (Catlin and H.-J. Lai [6]). Let G be a connected reduced graph of order n. Then $|E(G)| = 2n - 4$ if and only if $G = K_{2,n-2}$. $\Box$

4. **Main Result and Consequences**

The set of natural numbers is denoted by $\mathbf{N}$. Let K be a graph. A graph G is called K-free if it contains no subgraph K.

Here is our main result:

Theorem 1. Let n, m and p be natural numbers, $m, p \geq 2$. Let G be a 2-edge-connected simple graph of order n with $cl(G) = m$. If

$$|E(G)| \geq t(m, n-p+1) + 2p - 4, \tag{5}$$

then exactly one of the following holds:

(a) The reduction of G has order less than p;

(b) Equality holds in (5), $p \geq 4$ and G contains a subgraph $H = H_{m,n-p+1}$ such that the reduction of G is $G' = G/H = K_{2,p-2}$;

(c) $cl(G) = 3$, $n = p+3$, $p \geq 3$ and G contains a subgraph $H = K_3$ such that $G' = G/H = K_{2,p-1}$;

(d) G is a reduced graph with order n such that $n \geq 4$ and $n \in \{p+1, p+2, p+3, p+4, p+5, p+6\}$ and

$$2n - 4 \geq |E(G)| \geq \begin{cases} 2n-4 & \text{if } n = 6+p; \\ 2n-5 & \text{if } n = 5+p; \\ 2n-6 & \text{if } n = i+p, i \in \{2,3,4\}; \\ 2n-5 & \text{if } n = 1+p. \end{cases}$$

Note that $K_{2,c-2}$ is supereulerian if c is even. If $n > p+6$ then conclusions (c) and (d) of Theorem 1 are precluded. Hence, by Theorem F (b) we have following easy corollary:

Corollary 1. Let n, m and p be natural numbers, $m, p \geq 2$. Let G be a 2-edge-connected simple graph of order $n > p+6$ with $cl(G) = m$. If

$$|E(G)| \geq t(m, n-p+1) + 2p - 4, \tag{6}$$

then exactly one of the following holds:

(a) G is supereulerian;

(b) The reduction of G is a nonsupereulerian graph of order less than p;

(b) p is an odd number and equality holds in (6) and G contains a subgraph $H = H_{m,n-p+1}$ such that the reduction of G is $G' = G/H = K_{2,p-2}$. $\square$

In the following, we state some consequences of Theorem 1 first. The proof of Theorem 1 is given in the next section.

Corollary 2. Let G be a 2-edge-connected simple graph on n vertices, and let $p \in N - \{1\}$. If $cl(G) = m \geq 3$ and if

$$|E(G)| \geq t(m, n-p+1) + 2p - 4, \tag{7}$$

then exactly one of the following holds;

(a) The reduction of G has order less than p;

(b) Equality holds in (7) and G contains a subgraph $H = T_{m,n-p+1}$ such that the reduction of G is $G' = G/H = K_{2,p-2}$.

(c) $cl(G) = 3$ and $n = p + 3$ and G contains a $H = K_3$ such that the reduction of G is $G' = G/H = K_{2,p-1}$.

Proof. Let G be a graph satisfying the hypothesis of Corollary 2. Then G is not reduced since $cl(G) \geq 3$, and so (d) and (e) of Theorem 1 are precluded. It follows from Theorem 1 that the conclusion of Corollary 2 holds. □

Corollary 3. Let G be a 3-edge-connected simple graph of order n, and G' the reduction of G. If

$$|E(G)| \geq t(2, n - p + 1) + 2p - 4,$$

then exactly one of the following holds:

(a) G is collapsible;

(b) $1 < |V(G')| < p$.

(c) G is a reduced graph of order n such that $n \in \{p + 1, p + 2, p + 3, p + 4, p + 5\}$ and

$$2n - 5 \geq |E(G)| \geq \begin{cases} 2n - 5 & \text{if } n = 5 + p; \\ 2n - 6 & \text{if } n = i + p, i \in \{2, 3, 4\}; \\ 2n - 5 & \text{if } n = 1 + p. \end{cases}$$

Proof. Suppose that (a) fails. Then by Theorem F(a) $|V(G')| > 1$. By the definition of contraction, $\kappa'(G') \geq \kappa'(G) \geq 3$. Therefore, $G' \neq K_{2,c-2}$. The conclusions (b) and (c) of Theorem 1 are precluded. If Theorem 1(a) holds then $|V(G')| < p$ and so (b) of the corollary holds. Suppose that Theorem 1(d) holds. By Theorem G the case $|E(G)| = 2n - 4$ is impossible, and so (c) of the corollary holds.□

Corollary 4. Let G be a 2-edge-connected simple K_3-free graph of order n and let $p \in \mathbf{N} - \{1\}$. If

$$|E(G)| \geq t(2, n - p + 1) + 2p - 4, \tag{8}$$

then exactly one of the following holds:

(a) The reduction of G has order less than p;

(b) Equality holds in (8) and G contains a subgraph $H = T_{2,n-p+1}$ such that the reduction of G is $G' = G/H = K_{2,p-2}$;

(c) G is a reduced graph of order n such that $n \in \{p+1, p+2, p+3, p+4, p+5, p+6\}$ and

$$2n-4 \geq |E(G)| \geq \begin{cases} 2n-4 & \text{if } n = 6+p; \\ 2n-5 & \text{if } n = 5+p; \\ 2n-6 & \text{if } n = i+p, i \in \{2,3,4\}; \\ 2n-5 & \text{if } n = 1+p. \end{cases}$$

Proof. Since G is K_3-free, $cl(G) = m = 2$. Then the conclusion (c) of Theorem 1 are precluded. Note that the inequality (8) is a special case of (5) with $m = 2$ in Theorem 1. Obviously, Corollary 4 follows from Theorem 1. □

Corollary 5 (Catlin and Chen [5]). Let G be a 2-edge-connected simple graph of order n and let $p \in \mathbf{N} - \{1\}$. If

$$|E(G)| \geq \binom{n-p+1}{2} + 2p - 4, \tag{9}$$

then exactly one of these holds:

(a) The reduction of G has order less than p;

(b) Equality holds in (9), G has a complete subgraph H of order $n-p+1$, and the reduction of G is $G' = G/H = K_{2,p-2}$.

(c) G is a reduced graph such that either

$$|E(G)| \in \{2n-4, 2n-5\} \text{ and } n \in \{p+1, p+2\}$$

or

$$|E(G)| = 2n-4 \text{ and } n = p+3.$$

Proof. Choose m in Theorem 1 so that $m \geq n-p+1$. Then (5) and (4) together imply (9). Note that $m \geq n-p+1$ implies that $H_{m,n-p+1} = K_{n-p+1}$. Since $m \geq n-p+1$, (c) of Theorem 1 is impossible.

If (d) of Theorem 1 holds then G is a reduced graph with order $n \geq p+1$. By Theorem F(c) and (9),

$$2n-4 \geq |E(G)| \geq \binom{n-p+1}{2} + 2p - 4.$$

Then

$$4(n-p) \geq (n-p)(n-p+1).$$

Since $n \geq p+1$, we get $p+3 \geq n \geq p+1$. By (9) and routine computation, we can see that (c) of Corollary 5 holds. $\Box$

Remark. The case $p = 5$ of Corollary 3 is Theorem D which is a main result of Cai [3]. The case $p = 10$ of Corollary 3 for 3-edge-connected graph is Theorem E (Catlin and Chen [5]), which was a conjecture of Cai [3]. By (4), one can see that if $cl(G) = m < n - p + 1$ then inequalities in Corollaries 2, 3, and 4 have better lower bound than inequality (9) in Corollary 5. In the following we give some more results which improve the lower bounds of the inequalities in Theorem C and Theorem D.

We shall make use of the following lemma:

Lemma 1 (Chen [6]). Let G be a 3-edge-connected simple graph on $n \leq 11$ vertices. Then either G is collapsible or G is the Petersen graph. $\Box$

Corollary 6. Let G be a 2-edge connected simple graph of order n, and $cl(G) = m \geq 3$. If

$$|E(G)| \geq t(m, n-4) + 6, \tag{10}$$

then exactly one of the following holds:

(a) G is supereulerian;

(b) Equality holds in (10) and G has a subgraph $H = H_{m,n-4}$ such that the reduction of G is $G' = G/H = K_{2,3}$.

Proof. Set $p = 5$ in Corollary 2. Let G' be the reduction of G. If conclusion (a) of Corollary 2 holds, then G' has order at most 4. Note that any 2-edge-connected simple graph of order at most 4 are supereulerian, and so G' is supereulerian in this case. If (c) of Corollary 2 holds, then the reduction G' of G is $K_{2,4}$, which is also a supereulerian graph. By Theorem F(b), we can see that conclusion (a) of Corollary 4 holds if (a) or (c) of Corollary 2 holds.

If conclusion (b) of Corollary 2 holds, then G' is a nonsupereulerian graph $K_{2,3}$, and so (b) of the corollary holds. $\Box$

Corollary 7. Let G be a 3-edge-connected simple graph of order n with $cl(G) = m \geq 3$. If

$$|E(G)| \geq t(m, n-9) + 16, \tag{11}$$

then G is collapsible.

Proof. Set $p = 10$ in Corollary 3. Since $cl(G) \geq 3$, conclusion (c) of Corollary 3 is precluded. Let G' be the reduction of G. Suppose that G is not collapsible. Then (b) of Corollary 3 holds, and so G' has order less than $p = 10$. By Lemma 1, G' is collapsible, and so by Theorem F(a) $G' = K_1$, a contradiction. This proves the corollary. $\Box$

Corollary 8. Let G be a 2-edge-connected simple K_3-free graph of order n. If $n \geq 12$ and

$$|E(G)| \geq t(2, n-4) + 6, \tag{12}$$

then exactly one of the following holds:

(a) G is supereulerian;

(b) Equality holds in (12) and G contains a $H = T_{2,n-4}$ such that the reduction of G is $G' = G/H = K_{2,3}$.

Proof. Set $p = 5$ in of Corollary 4. Since $n \geq 12 = p + 7$, (c) of Corollary 4 is impossible. Note that any 2-edge-connected simple graph on $c \leq 4$ vertices is supereulerian. By Corollary 4, the statement follows. $\Box$.

Corollary 9. Let G be a 3-edge-connected simple K_3-free graph on n vertices. If $n \geq 16$ and

$$|E(G)| \geq t(2, n-9) + 16,$$

then G is collapsible.

Proof. Set $p = 10$ in of Corollary 3. Conclusion (c) of Corollary 3 is precluded by the hypothesis $n \geq 16$. Let G' be the reduction of G. Suppose that G is not collapsible. Then (b) of Corollary 3 holds, i.e., $1 < |V(G')| < 10$. Since $\kappa'(G') \geq \kappa'(G) \geq 3$, by Lemma 1, G' is collapsible. By Theorem F(a) $G' = K_1$, a contradiction. $\Box$

Remark. Let G be the simple graph obtained from the Petersen graph and the complete m-partite graph $T_{m,n-9}$ by identifying one vertex from each graph. Then G has order $n = (n-9) + 10 - 1$, and G is 3-edge-connected. The size of G is

$$|E(G)| = t(m, n-9) + 15.$$

Since the reduction of G is the Petersen graph, G is not collapsible. Hence, (11) and (13) are sharp.

5. **The Proof of Theorem 1**

Proof of Theorem 1. Let G' be the reduction of G and let $|V(G')| = c$. If $c = 1$ then G is collapsible and (a) of Theorem 1 holds. Suppose that $c > 1$ i.e., $G' \neq K_1$. Since G is 2-edge-connected and by the definition of contraction, we have $\kappa'(G') \geq \kappa'(G) \geq 2$. By Theorem F(c), G' is K_3-free, and so

$$c \geq 4, \tag{13}$$

and

$$|E(G')| \leq 2c - 4. \tag{14}$$

Let $V(G') = \{v_1, v_2, \cdots, v_c\}$, and let $H_1, H_2, \cdots, H_c$ be the preimages of $v_i's$ $(1 \leq i \leq c)$. Suppose that G has the maximum size among all K_{m+1}-free graphs which have the reduction G'. Then at most one H_i $(1 \leq i \leq c)$ is a nontrivial subgraph of G. Since G is K_{m+1}-free, this H_i is also K_{m+1}-free subgraph on $n-c+1$ vertices. Therefore, by the remark following Theorem E and (14)

$$\begin{aligned} |E(G)| &\leq |E(H_i)| + |E(G')| \\ &\leq t(m, n-c+1) + 2c - 4, \end{aligned} \tag{15}$$

with equality only if G has at most one subgraph H_i and it is a complete m-partite graph of order $n-c+1$, and its reduction graph G' has size $2c-4$.
By (5) and (15)

$$t(m, n-p+1) + 2p - 4 \leq |E(G)| \leq t(m, n-c+1) + 2c - 4, \tag{16}$$

and so

$$t(m, n-p+1) + 2p \leq t(m, n-c+1) + 2c. \tag{17}$$

Define $l(x) = \left\lfloor \frac{n-x+1}{m} \right\rfloor$. Then by (17) and the definition of $t(m, n-x+1)$ $(x = p$ or $c)$,

$$\begin{aligned} 2p \;&+ \binom{n-p+1-l(p)}{2} + (m-1)\binom{l(p)+1}{2} \\ \leq\;& 2c + \binom{n-c+1-l(c)}{2} + (m-1)\binom{l(c)+1}{2}, \end{aligned}$$

and so

$$\begin{aligned} &\binom{n-p+1-l(p)}{2} - \binom{n-c+1-l(c)}{2} \\ &+(m-1)\left\{\binom{l(p)+1}{2} - \binom{l(c)+1}{2}\right\} \leq 2(c-p). \end{aligned}$$

Simplifying the inequality above, we have the following

$$\begin{aligned} &\{c - p - (l(p) - l(c))\}(2n - p - c - l(p) - l(c) + 1) + \\ &\quad +(m-1)(l(p) - l(c))(l(p) + l(c) + 1) \leq 4(c-p). \end{aligned} \tag{18}$$

If $c < p$, then (a) of Theorem 1 holds. If $c = p$, then equality holds throughout (16). Therefore, $|E(G')| = 2c - 4 = 2p - 4$ in this case. By Theorem G, $G' = K_{2,p-2}$. By (13), $p \geq 4$. Thus (b) of Theorem 1 holds.

Next we consider the case

$$c > p.$$

<u>Case A</u> $m \geq n - p + 1$.

If $m = n - p + 1$ then $l(p) = 1$ and $l(c) = 0$ since $c > p$. If $m > n - p + 1$ then $l(p) = l(c) = 0$. By (18), we have that in either case

$$2n \leq c + p + 3.$$

If $c < n$, then $n \geq c+2$ since G cannot have its reducton of order $n-1$. Hence $n \leq p+1 \leq c$, a contradiction. It follows that $n = c$. Then G is reduced, and so $m = 2$. Then

$$p < n \leq p + m - 1 = p + 1. \tag{19}$$

Since G is reduced, (14) gives $2n - 4 \geq |E(G)|$. By (13) $n = c \geq 4$. By (5) and routine computation, we have

$$2n - 4 \geq |E(G)| \geq 2n - 5 \quad \text{if } n = p + 1,$$

and so (d) of Theorem 1 holds.

<u>Case B</u> $m < n - p + 1$.

By the definition of $l(p)$ and $l(c)$, we have that $n - p + 1 = l(p)m + r_p$ and $n - c + 1 = l(c)m + r_c$ for some $r_p, r_c \in \{0, 1, 2, \cdots, m - 1\}$. Then

$$\begin{aligned} l(p) - l(c) &= \frac{n-p+1}{m} - \frac{r_p}{m} - \frac{n-c+1}{m} + \frac{r_c}{m} \\ &= \frac{c-p}{m} + \frac{r_c - r_p}{m}, \end{aligned} \tag{20}$$

and

$$l(p) + l(c) = \frac{2n-p-c+2}{m} - \frac{r_p + r_c}{m}, \tag{21}$$

where $r_p, r_c \in \{0, 1, 2, \cdots, m - 1\}$.

By (18), (20) and (21),

$$\begin{aligned} &(c - p - \frac{c-p}{m} - \frac{r_c - r_p}{m})(2n - p - c - \frac{2n-p-c+2}{m} + \frac{r_c + r_p}{m} + 1) \\ &+(m-1)(\frac{c-p}{m} + \frac{r_c - r_p}{m})(\frac{2n-p-c+2}{m} - \frac{r_c + r_p}{m} + 1) \\ &\leq 4(c - p). \end{aligned} \tag{22}$$

Simplifying the inequality (22), we have the following

$$(1-\frac{1}{m})(c-p)(2n-p-c+2) \quad - \quad \frac{(r_c-r_p)(r_c+r_p-m)}{m} \\ \leq \quad 4(c-p). \tag{23}$$

Since $c > p$, and by (23)

$$(2n-p-c+2) \leq \frac{4m}{m-1} + \frac{(r_c-r_p)(r_c+r_p-m)}{(m-1)(c-p)}, \tag{24}$$

where $r_p, r_c \in \{0, 1, 2, \cdots, m-1\}$.

Consider the function $f(x, y) = x^2 - y^2 - m(x-y)$ on domain $D = \{(x, y)|0 \leq x \leq m-1, 0 \leq y \leq m-1\}$. Note that the maximum value of $f(x, y)$ can be obtained on the boundery of its domain. It is routine to check that

$$\max_{(x,y)\in D} f(x, y) = f(0, \frac{m}{2}) = \frac{m^2}{4}.$$

Hence, we have that

$$f(r_c, r_p) = (r_c-r_p)(r_c+r_p-m) \leq \frac{m^2}{4}. \tag{25}$$

By (24) and (25)

$$2n-c-p+2 \leq \frac{4m}{m-1} + \frac{m^2}{4(m-1)(c-p)}, \tag{26}$$

and so

$$2n \leq 2+c+p+\frac{4}{m-1}+\frac{m}{4(c-p)}+\frac{1}{4(c-p)}+\frac{1}{4(c-p)(m-1)}. \tag{27}$$

<u>Subcase B1</u> Suppose that $c < n$. Since G is simple, G cannot have its reduction of order $n-1$. Hence,

$$n \geq c+2. \tag{28}$$

If $m = 2$, then G is K_3-free. By (27)

$$2n \leq 6+p+c+\frac{1}{c-p}.$$

Since $p+1 \leq c$, by (28), we have

$$n \leq 4+p+\frac{1}{c-p} \leq 4+p+1 \leq 4+c. \tag{29}$$

But in this case G is simple and K_3-free, and so G has no nontrivial collapsible subgraph of order less than 6. Hence, the reduction of G cannot have order $c \geq n-4$, contrary to inequality (29).

If $m \geq 3$ and G has a complete subgraph K_m then $c \leq |V(G/K_m)|$. If follows that in this case we have

$$c \leq |V(G/K_m)| = n - m + 1. \tag{30}$$

By (27), (28) and (30),

$$n \leq p + 3 - m + \frac{4}{m-1} + \frac{m}{4(c-p)} + \frac{1}{4(c-p)} + \frac{1}{4(c-p)(m-1)}. \tag{31}$$

If $m \geq 4$ then by $c \geq p+1$ and (30),

$$p + 4 = (p+1) + 4 - 1 \leq c + m - 1 \leq n.$$

From another way, by (31) and $c - p \geq 1$,

$$\begin{aligned} n &\leq p + 3 - m + \frac{4}{3} + \frac{m}{4} + \frac{1}{4} + \frac{1}{12}, \\ n &\leq p + 3 - \frac{3}{4}m + \frac{5}{3}, \\ n &\leq p + 3 - \frac{3}{4}(4) + \frac{5}{3} = p + \frac{5}{3}, \end{aligned}$$

a contradiction.

If $m = 3$, then by (28) and $c \geq p+1$, we have $n \geq 3 + p$. Hence $n = p + 3$, and so $c = n - 2$. This shows that G contains a triangle $H = K_3$ such that $G' = G/H$ on $p+1$ vertices and

$$|E(G')| = |E(G)| - 3.$$

As a special case of (16), we have that

$$t(3, n-p+1) + 2p - 4 \leq |E(G)| \leq t(3, n-c+1) + 2c - 4,$$

and so,

$$t(3,4) + 2(n-3) - 4 \leq |E(G)| \leq t(3,3) + 2(n-2) - 4.$$

Therefore,

$$|E(G)| = 2n - 5.$$

Hence,

$$|E(G')| = |E(G)| - 3 = (2n-5) - 3 = 2(n-2) - 4 = 2c - 4.$$

By Theorem G and $c = p + 1$, $G' = K_{2,c-2} = K_{2,p-1}$. By (13), $p = c - 1 \geq 3$ and so (c) of Theorem 1 holds.

Subcase B2 $c = n$. Then by (13) $n \geq 4$ and G is a reduced graph. By Theorem F(c) G is K_3-free. Hence $m = 2$. By (14)

$$|E(G)| \leq 2n - 4. \tag{32}$$

By (31),

$$n \leq 2 + p + 4 + \frac{1}{n-p}. \tag{33}$$

If $n = p + 1$ then by the hypothesis of Case B, $2 = m < n - p + 1 = 2$, a contradiction.

If $n \geq p + 2$. Then by (33),

$$p + 2 \leq n \leq 2 + p + 4 + \frac{1}{2}. \tag{34}$$

$$p + 2 \leq n \leq 6 + p. \tag{35}$$

By (35), (5) and routine computation, we have the following;

$$2n - 4 \geq |E(G)| \geq \begin{cases} 2n-4 & \text{if } n = 6+p; \\ 2n-5 & \text{if } n = 5+p; \\ 2n-6 & \text{if } n = i+p, i \in \{2,3,4\}; \end{cases}$$

The conclusion (d) of Theorem 1 holds.

The proof of Theorem 1 is complete. □

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