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On extremal nonsupereulerian graphs
with clique number m

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Abstract

A graph $G$ is supereulerian if it contains a spanning eulerian subgraph. Let $n$, $m$ and $p$ be natural numbers, $m, p \geq 2$. Let $G$ be a 2-edge-connected simple graph on $n > p + 6$ vertices containing no $K_{m+1}$. We prove that if

$$|E(G)| \geq \left( \frac{n-p+1-k}{2} \right) + (m-1) \left( \frac{k+1}{2} \right) + 2p - 4,$$

where $k = \lfloor \frac{n-p+1}{m} \rfloor$, then either $G$ is supereulerian, or $G$ can be contracted to a nonsupereulerian graph of order less than $p$, or equality holds in (1) and $G$ can be contracted to $K_{2,p-2}$ ($p$ is odd) by contracting a complete $m$-partite graph $T_{m,n-p+1}$ of order $n-p+1$ in $G$. This is a generalization of the previous results in [3] and [5].

1. Introduction

We follow the notation of Bondy and Murty [1], except that graphs have no loops. For a graph $G$, the order of the maximum complete subgraph of $G$ is called clique number of $G$ and denoted by $cl(G)$. A graph is eulerian if it is connected and every vertex has even degree. A graph $G$ is called supereulerian if it has a spanning eulerian subgraph $H$. A cycle $C$ of $G$ is called a hamiltonian cycle if $V(C) = V(G)$ and is called dominating cycle if $E(G - V(C)) = \emptyset$. A graph is hamiltonian if it contains a hamiltonian cycle. Obviously, hamiltonian graphs are special supereulerian graphs.

There is rich literature on the following extremal graph theory problems: for a given family $\mathcal{F}$ of graphs and for a natural number $n$, what is the maximum size of simple graphs of order $n$ which are not in $\mathcal{F}$. For example, when $\mathcal{F} = \{\text{graphs with clique number at least } m\}$, this is Turán’s Theorem. In this note, we consider the family

$\mathcal{F} = \{\text{supereulerian graphs with clique number } m\}$.

In fact, our results are related to Turán’s Theorem.
Let $G$ be a graph, and let $H$ be a connected subgraph of $G$. The contraction $G/H$ is the graph obtained from $G$ by contracting all edges of $H$, and by deleting any resulting loops. Even when $G$ is simple, $G/H$ may not be.

Here are some prior results related to our subject.

**Theorem A** (Ore [8] and Bondy [2]). Let $G$ be a simple graph on $n$ vertices. If
\[
|E(G)| \geq \left( \frac{n-1}{2} \right) + 2,
\]
then exactly one of the following holds:
(a) $G$ is hamiltonian;
(b) Equality holds in (2), and $G \in \{K_1 \lor (K_1 + K_{n-2}), K_2 + K_{n-2}^c\}$ (where $K_{n-2}^c$ is the complement of $K_3$). \(\square\)

**Theorem B** (Veldman [10]). Let $G$ be a 2-edge-connected simple graph of order $n$. If
\[
|E(G)| \geq \left( \frac{n-4}{2} \right) + 11,
\]
then $G$ has a dominating cycle. \(\square\)

**Theorem C** (Cai [3]). Let $G$ be 2-edge-connected simple graph on $n$ vertices. If
\[
|E(G)| \geq \left( \frac{n-4}{2} \right) + 6,
\]
then exactly one of the following holds:
(a) $G$ is supereulerian;
(b) $G = K_{2,5}$;
(c) Equality holds in (3), and either $G = Q_3 - v$ (the cube minus a vertex), or $G$ contains a complete subgraph $H = K_{n-4}$ such that $G/H = K_{2,3}$. \(\square\)

**Theorem D** (Catlin and Chen [5]). Let $G$ be a 3-edge-connected simple graph on $n$ vertices. If
\[
|E(G)| \geq \left( \frac{n-9}{2} \right) + 16,
\]
then $G$ is supereulerian. \(\square\)

In this paper, following closely the method of [5], we shall generalize Theorem C and Theorem D. In particular, we found that if a graph $G$ is $K_3$-free or has small clique number then the lower bound of the inequalities in Theorem C and Theorem D can be improved.
2. Notation and Turán’s Theorem

Let \( n \) and \( m \) be natural numbers, we define \( t(m, n) \) as the following:

\[
t(m, n) = \left( n - \frac{k}{2} \right) + (m - 1) \left( \frac{k + 1}{2} \right),
\]

where \( k = \lfloor \frac{n}{m} \rfloor \). It is easy to see that if \( m = n \) or \( m > n \) then \( k = 1 \) or \( k = 0 \), respectively, and so the right side of the equation above is equal to \( \frac{n}{2} \). If \( m = 2 \) then

\[
t(2, n) = \begin{cases} 
\frac{n^2}{4} & \text{if } n \text{ is even;} \\
\frac{n^2 - 1}{4} & \text{if } n \text{ is odd.}
\end{cases}
\]

Note that for \( m > n \),

\[
t(2, n) < t(3, n) < \cdots < t(n - 1, n) < t(n, n) = t(m, n) = \left( \frac{n}{2} \right).
\]

One can see that \( t(m, n) \) is related to the Turán numbers below.

For \( m \leq n \), denote by \( T_{m, n} \) the complete \( m \)-partite graph of order \( n \) with \( \left\lfloor \frac{n}{m} \right\rfloor, \left\lfloor \frac{n + 1}{m} \right\rfloor, \ldots, \left\lfloor \frac{n + m - 1}{m} \right\rfloor \) vertices in the various independent classes. Note that \( T_{m, n} \) is the unique complete \( m \)-partite graph of order \( n \) whose independent classes are as equal as possible and \( T_{n, n} = K_n \). Let \( k = \lfloor \frac{n}{m} \rfloor \), it is known that the size of \( T_{m, n} \) is

\[
|E(T_{m, n})| = t(m, n) = \left( n - \frac{k}{2} \right) + (m - 1) \left( \frac{k + 1}{2} \right).
\]

**Theorem E** (Turán [9]). Let \( m \) and \( n \) be natural numbers, \( m \geq 2 \). Then every graph of order \( n \) and size greater than \( |E(T_{m, n})| \) contains a \( K_{m+1} \). Furthermore, \( T_{m, n} \) is the only graph of order \( n \) and size \( |E(T_{m, n})| \) that does not contain a \( K_{m+1} \). \( \Box \)

**Remark.** Let \( G \) be a graph of order \( n \) with maximum size that does not contain a \( K_{m+1} \). If \( m > n \) then \( |E(G)| = \left( \frac{n}{2} \right) \). If \( m \leq n \) then by Theorem E \( |E(G)| \leq |E(T_{m, n})| \). Thus, if \( G \) is a graph containing no \( K_{m+1} \) then \( |E(G)| \leq t(m, n) \). For convenience, we define

\[
H_{m, n} = \begin{cases} 
T_{m, n} & \text{if } m < n; \\
K_n & \text{if } m \geq n.
\end{cases}
\]
3. Catlin’s Reduction Method

The following concept was given by Catlin [4].

For a graph $G$, let $O(G)$ denoted the set of vertices of odd degree in $G$. A graph $G$ is called collapsible if for every even set $X \subseteq V(G)$ there is a spanning connected subgraph $H_X$ of $G$, such that $O(H_X) = X$. The trivial graph $K_1$ is both supereulerian and collapsible. The cycles $C_2$ and $C_3$ are collapsible, but $C_t$ is not if $t \geq 4$. In fact, if $G$ is collapsible then $G$ contains a spanning $(u, v)$-trail for any $u, v \in V(G)$. In particular, a collapsible graph is supereulerian.

In [4], Catlin showed that every graph $G$ has a unique collection of disjoint maximal collapsible subgraphs $H_1, H_2, \ldots, H_c$. Define $G'$ to be the graph obtained from $G$ by contracting each $H_i$ into a single vertex, $(1 \leq i \leq c)$. Since $V(G) = V(H_1) \cup \cdots \cup V(H_c)$, the graph $G'$ has order $c$. We call the graph $G'$ the reduction of $G$. Any graph $G$ has a unique reduction $G'[4]$. A graph $G$ is reduced if $G = G'$.

We shall make use of the following theorems:

**Theorem F** (Catlin [4]) Let $G$ be a graph. Let $G'$ be the reduction of $G$.

(a) Let $H$ be a collapsible subgraph of $G$. Then $G$ is collapsible if and only if $G/H$ is collapsible. In particular, $G$ is collapsible if and only if $G' = K_1$.

(b) $G$ is supereulerian if and only if $G'$ is supereulerian.

(c) If $G$ is a reduced graph of order $n$, then $G$ is simple and $K_3$-free with $\delta(G) \leq 3$ and either $G \in \{K_1, K_2\}$, or $|E(G)| \leq 2n - 4$. \hfill $\square$

**Theorem G** (Catlin and H.-J. Lai [6]). Let $G$ be a connected reduced graph of order $n$. Then $|E(G)| = 2n - 4$ if and only if $G = K_{2, n-2}$. \hfill $\square$

4. Main Result and Consequences

The set of natural numbers is denoted by $\mathbb{N}$. Let $K$ be a graph. A graph $G$ is called $K$-free if it contains no subgraph $K$.

Here is our main result:

**Theorem 1.** Let $n, m$ and $p$ be natural numbers, $m, p \geq 2$. Let $G$ be a 2-edge-connected
simple graph of order \( n \) with \( cl(G) = m \). If
\[
|E(G)| \geq t(m, n - p + 1) + 2p - 4,
\]
then exactly one of the following holds:
(a) The reduction of \( G \) has order less than \( p \);
(b) Equality holds in (5), \( p \geq 4 \) and \( G \) contains a subgraph \( H = H_{m,n-p+1} \) such that the reduction of \( G \) is \( G' = G/H = K_{2,p-2} \);
(c) \( cl(G) = 3, n = p + 3, p \geq 3 \) and \( G \) contains a subgraph \( H = K_3 \) such that \( G' = G/H = K_{2,p-1} \);
(d) \( G \) is a reduced graph with order \( n \) such that \( n \geq 4 \) and \( n \in \{p + 1, p + 2, p + 3, p + 4, p + 5, p + 6\} \) and
\[
2n - 4 \geq |E(G)| \geq \begin{cases} 
2n - 4 & \text{if } n = 6 + p; \\
2n - 5 & \text{if } n = 5 + p; \\
2n - 6 & \text{if } n = i + p, i \in \{2, 3, 4\}; \\
2n - 5 & \text{if } n = 1 + p.
\end{cases}
\]

Note that \( K_{2,c-2} \) is supereulerian if \( c \) is even. If \( n > p + 6 \) then conclusions (c) and (d) of Theorem 1 are precluded. Hence, by Theorem F (b) we have following easy corollary:

**Corollary 1.** Let \( n, m \) and \( p \) be natural numbers, \( m, p \geq 2 \). Let \( G \) be a 2-edge-connected simple graph of order \( n > p + 6 \) with \( cl(G) = m \). If
\[
|E(G)| \geq t(m, n - p + 1) + 2p - 4,
\]
then exactly one of the following holds:
(a) \( G \) is supereulerian;
(b) The reduction of \( G \) is a nonsupereulerian graph of order less than \( p \);
(b) \( p \) is an odd number and equality holds in (6) and \( G \) contains a subgraph \( H = H_{m,n-p+1} \) such that the reduction of \( G \) is \( G' = G/H = K_{2,p-2} \). \( \square \)

In the following, we state some consequences of Theorem 1 first. The proof of Theorem 1 is given in the next section.

**Corollary 2.** Let \( G \) be a 2-edge-connected simple graph on \( n \) vertices, and let \( p \in N \setminus \{1\}. \) If \( cl(G) = m \geq 3 \) and if
\[
|E(G)| \geq t(m, n - p + 1) + 2p - 4,
\]
then exactly one of the following holds;
(a) The reduction of $G$ has order less than $p$;
(b) Equality holds in (7) and $G$ contains a subgraph $H = T_{m,n-p+1}$ such that the reduction of $G$ is $G' = G/H = K_{2,p-2}$.
(c) $cl(G) = 3$ and $n = p + 3$ and $G$ contains a $H = K_3$ such that the reduction of $G$ is $G' = G/H = K_{2,p-1}$.

**Proof.** Let $G$ be a graph satisfying the hypothesis of Corollary 2. Then $G$ is not reduced since $cl(G) \geq 3$, and so (d) and (e) of Theorem 1 are precluded. It follows from Theorem 1 that the conclusion of Corollary 2 holds. □

**Corollary 3.** Let $G$ be a 3-edge-connected simple graph of order $n$, and $G'$ the reduction of $G$. If

$$|E(G)| \geq t(2, n-p+1) + 2p - 4,$$

then exactly one of the following holds:

(a) $G$ is collapsible;
(b) $1 < |V(G')| < p$.
(c) $G$ is a reduced graph of order $n$ such that $n \in \{p + 1, p + 2, p + 3, p + 4, p + 5\}$ and

$$2n - 5 \geq |E(G)| \geq \begin{cases} 
2n - 5 & \text{if } n = 5 + p; \\
2n - 6 & \text{if } n = i + p, i \in \{2, 3, 4\}; \\
2n - 5 & \text{if } n = 1 + p.
\end{cases}$$

**Proof.** Suppose that (a) fails. Then by Theorem F(a) $|V(G')| > 1$. By the definition of contraction, $\kappa'(G') \geq \kappa'(G) \geq 3$. Therefore, $G' \neq K_{2,c-2}$. The conclusions (b) and (c) of Theorem 1 are precluded. If Theorem 1(a) holds then $|V(G')| < p$ and so (b) of the corollary holds. Suppose that Theorem 1(d) holds. By Theorem G the case $|E(G)| = 2n - 4$ is impossible, and so (c) of the corollary holds. □

**Corollary 4.** Let $G$ be a 2-edge-connected simple $K_3$-free graph of order $n$ and let $p \in N - \{1\}$. If

$$|E(G)| \geq t(2, n-p+1) + 2p - 4,$$

then exactly one of the following holds:

(a) The reduction of $G$ has order less than $p$;
(b) Equality holds in (8) and $G$ contains a subgraph $H = T_{2,n-p+1}$ such that the reduction of $G$ is $G' = G/H = K_{2,p-2}$;

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(c) $G$ is a reduced graph of order $n$ such that $n \in \{p+1, p+2, p+3, p+4, p+5, p+6\}$ and

$$2n - 4 \geq |E(G)| \geq \begin{cases} 
2n - 4 & \text{if } n = 6 + p; \\
2n - 5 & \text{if } n = 5 + p; \\
2n - 6 & \text{if } n = i + p, i \in \{2, 3, 4\}; \\
2n - 5 & \text{if } n = 1 + p. 
\end{cases}$$

**Proof.** Since $G$ is $K_3$-free, $cl(G) = m = 2$. Then the conclusion (c) of Theorem 1 are precluded. Note that the inequality (8) is a special case of (5) with $m = 2$ in Theorem 1. Obviously, Corollary 4 follows from Theorem 1. □

**Corollary 5** (Catlin and Chen [5]). Let $G$ be a 2-edge-connected simple graph of order $n$ and let $p \in \mathbb{N} - \{1\}$. If

$$|E(G)| \geq \binom{n - p + 1}{2} + 2p - 4,$$  \hspace{1cm} (9)$$

then exactly one of these holds:

(a) The reduction of $G$ has order less than $p$;
(b) Equality holds in (9), $G$ has a complete subgraph $H$ of order $n-p+1$, and the reduction of $G$ is $G' = G/H = K_2,p-2$.
(c) $G$ is a reduced graph such that either

$$|E(G)| \in \{2n - 4, 2n - 5\} \text{ and } n \in \{p + 1, p + 2\}$$

or

$$|E(G)| = 2n - 4 \text{ and } n = p + 3.$$  \hspace{1cm} (10)$$

**Proof.** Choose $m$ in Theorem 1 so that $m \geq n - p + 1$. Then (5) and (4) together imply (9). Note that $m \geq n - p + 1$ implies that $H_{m,n-p+1} = K_{n-p+1}$. Since $m \geq n - p + 1$, (c) of Theorem 1 is impossible.

If (d) of Theorem 1 holds then $G$ is a reduced graph with order $n \geq p + 1$. By Theorem F(c) and (9),

$$2n - 4 \geq |E(G)| \geq \binom{n - p + 1}{2} + 2p - 4.$$

Then

$$4(n - p) \geq (n - p)(n - p + 1).$$
Since \( n \geq p + 1 \), we get \( p + 3 \geq n \geq p + 1 \). By (9) and routine computation, we can see that (c) of Corollary 5 holds. □

**Remark.** The case \( p = 5 \) of Corollary 3 is Theorem D which is a main result of Cai [3]. The case \( p = 10 \) of Corollary 3 for 3-edge-connected graph is Theorem E (Catlin and Chen [5]), which was a conjecture of Cai [3]. By (4), one can see that if \( cl(G) = m < n - p + 1 \) then inequalities in Corollaries 2, 3, and 4 have better lower bound than inequality (9) in Corollary 5. In the following we give some more results which improve the lower bounds of the inequalities in Theorem C and Theorem D.

We shall make use of the following lemma:

**Lemma 1** (Chen [6]). Let \( G \) be a 3-edge-connected simple graph on \( n \leq 11 \) vertices. Then either \( G \) is collapsible or \( G \) is the Petersen graph. □

**Corollary 6.** Let \( G \) be a 2-edge connected simple graph of order \( n \), and \( cl(G) = m \geq 3 \). If
\[
|E(G)| \geq t(m, n - 4) + 6,
\]
then exactly one of the following holds:

(a) \( G \) is supereulerian;

(b) Equality holds in (10) and \( G \) has a subgraph \( H = H_{m,n-4} \) such that the reduction of \( G \) is \( G' = G/H = K_{2,3} \).

**Proof.** Set \( p = 5 \) in Corollary 2. Let \( G' \) be the reduction of \( G \). If conclusion (a) of Corollary 2 holds, then \( G' \) has order at most 4. Note that any 2-edge-connected simple graph of order at most 4 are supereulerian, and so \( G' \) is supereulerian in this case. If (c) of Corollary 2 holds, then the reduction \( G' \) of \( G \) is \( K_{2,4} \), which is also a supereulerian graph. By Theorem F(b), we can see that conclusion (a) of Corollary 4 holds if (a) or (c) of Corollary 2 holds.

If conclusion (b) of Corollary 2 holds, then \( G' \) is a nonsupereulerian graph \( K_{2,3} \), and so (b) of the corollary holds. □

**Corollary 7.** Let \( G \) be a 3-edge-connected simple graph of order \( n \) with \( cl(G) = m \geq 3 \). If
\[
|E(G)| \geq t(m, n - 9) + 16,
\]
then \( G \) is collapsible.

**Proof.** Set \( p = 10 \) in Corollary 3. Since \( cl(G) \geq 3 \), conclusion (c) of Corollary 3 is precluded. Let \( G' \) be the reduction of \( G \). Suppose that \( G \) is not collapsible. Then (b) of Corollary 3 holds, and so \( G' \) has order less than \( p = 10 \). By Lemma 1, \( G' \) is collapsible, and so by Theorem F(a) \( G' = K_1 \), a contradiction. This proves the corollary. □
Corollary 8. Let $G$ be a 2-edge-connected simple $K_3$-free graph of order $n$. If $n \geq 12$ and

$$|E(G)| \geq t(2, n-4) + 6,$$  \hfill (12)

then exactly one of the following holds:

(a) $G$ is supereulerian;
(b) Equality holds in (12) and $G$ contains a $H = T_{2, n-4}$ such that the reduction of $G$ is $G' = G/H = K_{2, 3}$.

Proof. Set $p = 5$ in of Corollary 4. Since $n \geq 12 = p + 7$, (c) of Corollary 4 is impossible. Note that any 2-edge-connected simple graph on $c \leq 4$ vertices is supereulerian. By Corollary 4, the statement follows. \(\square\)

Corollary 9. Let $G$ be a 3-edge-connected simple $K_3$-free graph on $n$ vertices. If $n \geq 16$ and

$$|E(G)| \geq t(2, n-9) + 16,$$

then $G$ is collapsible.

Proof. Set $p = 10$ in of Corollary 3. Conclusion (c) of Corollary 3 is precluded by the hypothesis $n \geq 16$. Let $G'$ be the reduction of $G$. Suppose that $G$ is not collapsible. Then (b) of Corollary 3 holds, i.e., $1 < |V(G')| < 10$. Since $\kappa'(G') \geq \kappa'(G) \geq 3$, by Lemma 1, $G'$ is collapsible. By Theorem F(a) $G' = K_1$, a contradiction. \(\square\)

Remark. Let $G$ be the simple graph obtained from the Petersen graph and the complete $m$-partite graph $T_{m, n-9}$ by identifying one vertex from each graph. Then $G$ has order $n = (n-9) + 10 - 1$, and $G$ is 3-edge-connected. The size of $G$ is

$$|E(G)| = t(m, n-9) + 15.$$

Since the reduction of $G$ is the Petersen graph, $G$ is not collapsible. Hence, (11) and (13) are sharp.

5. The Proof of Theorem 1

Proof of Theorem 1. Let $G'$ be the reduction of $G$ and let $|V(G')| = c$. If $c = 1$ then $G$ is collapsible and (a) of Theorem 1 holds. Suppose that $c > 1$ i.e., $G' \neq K_1$. Since $G$ is 2-edge-connected and by the definition of contraction, we have $\kappa'(G') \geq \kappa'(G) \geq 2$. By Theorem F(c), $G'$ is $K_3$-free, and so

$$c \geq 4,$$  \hfill (13)
and

\[ |E(G')| \leq 2c - 4. \quad (14) \]

Let \( V(G') = \{v_1, v_2, \ldots, v_c\} \), and let \( H_1, H_2, \ldots, H_c \) be the preimages of \( v'_i \)s \((1 \leq i \leq c)\). Suppose that \( G \) has the maximum size among all \( K_{m+1} \)-free graphs which have the reduction \( G' \). Then at most one \( H_i \) \((1 \leq i \leq c)\) is a nontrivial subgraph of \( G \). Since \( G \) is \( K_{m+1} \)-free, this \( H_i \) is also \( K_{m+1} \)-free subgraph on \( n - c + 1 \) vertices. Therefore, by the remark following Theorem E and (14)

\[ |E(G)| \leq |E(H_i)| + |E(G')| \leq t(m, n - c + 1) + 2c - 4, \quad (15) \]

with equality only if \( G \) has at most one subgraph \( H_i \) and it is a complete \( m \)-partite graph of order \( n - c + 1 \), and its reduction graph \( G' \) has size \( 2c - 4 \).

By (5) and (15)

\[ t(m, n - p + 1) + 2p - 4 \leq |E(G)| \leq t(m, n - c + 1) + 2c - 4, \quad (16) \]

and so

\[ t(m, n - p + 1) + 2p \leq t(m, n - c + 1) + 2c. \quad (17) \]

Define \( l(x) = \left\lceil \frac{n-x+1}{m} \right\rceil \). Then by (17) and the definition of \( t(m, n - x + 1) \) \((x = p \text{ or } c)\),

\[ 2p + \left( \frac{n-p+1-l(p)}{2} \right) + (m-1) \left( \frac{l(p)+1}{2} \right) \leq 2c + \left( \frac{n-c+1-l(c)}{2} \right) + (m-1) \left( \frac{l(c)+1}{2} \right), \]

and so

\[ \left( \frac{n-p+1-l(p)}{2} \right) - \left( \frac{n-c+1-l(c)}{2} \right) + (m-1) \left\{ \left( \frac{l(p)+1}{2} \right) - \left( \frac{l(c)+1}{2} \right) \right\} \leq 2(c-p). \]

Simplifying the inequality above, we have the following

\[ \{c - p - (l(p) - l(c))\}(2n - p - c - l(p) - l(c) + 1) + (m-1)(l(p) - l(c))(l(p) + l(c) + 1) \leq 4(c - p). \quad (18) \]
If \( c < p \), then (a) of Theorem 1 holds. If \( c = p \), then equality holds throughout (16). Therefore, \(|E(G')| = 2c - 4 = 2p - 4\) in this case. By Theorem G, \( G' = K_{2,p-2} \). By (13), \( p \geq 4 \). Thus (b) of Theorem 1 holds.

Next we consider the case

\[ c > p. \]

**Case A** \( m \geq n - p + 1 \).

If \( m = n - p + 1 \) then \( l(p) = 1 \) and \( l(c) = 0 \) since \( c > p \). If \( m > n - p + 1 \) then \( l(p) = l(c) = 0 \). By (18), we have that in either case

\[ 2n \leq c + p + 3. \]

If \( c < n \), then \( n \geq c + 2 \) since \( G \) cannot have its reduction of order \( n - 1 \). Hence \( n \leq p + 1 \leq c \), a contradiction. It follows that \( n = c \). Then \( G \) is reduced, and so \( m = 2 \). Then

\[ p < n \leq p + m - 1 = p + 1. \]  

(19)

Since \( G \) is reduced, (14) gives \( 2n - 4 \geq |E(G)| \). By (13) \( n = c \geq 4 \). By (5) and routine computation, we have

\[ 2n - 4 \geq |E(G)| \geq 2n - 5 \quad \text{if} \quad n = p + 1, \]

and so (d) of Theorem 1 holds.

**Case B** \( m < n - p + 1 \).

By the definition of \( l(p) \) and \( l(c) \), we have that \( n - p + 1 = l(p)m + r_p \) and \( n - c + 1 = l(c)m + r_c \) for some \( r_p, r_c \in \{0, 1, 2, \ldots, m - 1\} \). Then

\[
\begin{align*}
    l(p) - l(c) &= \frac{n - p + 1}{m} - \frac{r_p}{m} - \frac{n - c + 1}{m} + \frac{r_c}{m} \\
    &= \frac{c - p}{m} + \frac{r_c - r_p}{m}, \\
\end{align*}
\]

(20)

and

\[
\begin{align*}
    l(p) + l(c) &= \frac{2n - p - c + 2}{m} - \frac{r_p + r_c}{m}, \\
\end{align*}
\]

(21)

where \( r_p, r_c \in \{0, 1, 2, \ldots, m - 1\} \).

By (18), (20) and (21),

\[
\begin{align*}
    &\left(c - p - \frac{c - p}{m} - \frac{r_c - r_p}{m}\right)(2n - p - c - \frac{2n - p - c + 2}{m} + \frac{r_c + r_p}{m} + 1) \\
    &+ (m - 1)\left(c - p - \frac{c - p}{m} + \frac{r_c - r_p}{m}\right)(\frac{2n - p - c + 2}{m} - \frac{r_c + r_p}{m} + 1) \\
    &\leq 4(c - p). \\
\end{align*}
\]

(22)
Simplifying the inequality (22), we have the following

\[
(1 - \frac{1}{m})(c - p)(2n - p - c + 2) - \frac{(r_c - r_p)(r_c + r_p - m)}{m} \leq 4(c - p).
\]

(23)

Since \( c > p \), and by (23)

\[
(2n - p - c + 2) \leq \frac{4m}{m - 1} + \frac{(r_c - r_p)(r_c + r_p - m)}{(m - 1)(c - p)},
\]

(24)

where \( r_p, r_c \in \{0, 1, 2, \ldots, m - 1\} \).

Consider the function \( f(x, y) = x^2 - y^2 - m(x - y) \) on domain \( D = \{(x, y)|0 \leq x \leq m - 1, 0 \leq y \leq m - 1\} \). Note that the maximum value of \( f(x, y) \) can be obtained on the boundary of its domain. It is routine to check that

\[
\max_{(x,y) \in D} f(x, y) = f(0, \frac{m}{2}) = \frac{m^2}{4}.
\]

Hence, we have that

\[
f(r_c, r_p) = (r_c - r_p)(r_c + r_p - m) \leq \frac{m^2}{4}.
\]

(25)

By (24) and (25)

\[
2n - c - p + 2 \leq \frac{4m}{m - 1} + \frac{m^2}{4(m - 1)(c - p)},
\]

(26)

and so

\[
2n \leq 2 + c + p + \frac{4}{m - 1} + \frac{m}{4(c - p)} + \frac{1}{4(c - p)} + \frac{1}{4(c - p)(m - 1)}.
\]

(27)

**Subcase B1** Suppose that \( c < n \). Since \( G \) is simple, \( G \) cannot have its reduction of order \( n - 1 \). Hence,

\[
n \geq c + 2.
\]

(28)

If \( m = 2 \), then \( G \) is \( K_3 \)-free. By (27)

\[
2n \leq 6 + p + c + \frac{1}{c - p}.
\]

Since \( p + 1 \leq c \), by (28), we have

\[
n \leq 4 + p + \frac{1}{c - p} \leq 4 + p + 1 \leq 4 + c.
\]

(29)
But in this case $G$ is simple and $K_3$-free, and so $G$ has no nontrivial collapsible subgraph of order less than 6. Hence, the reduction of $G$ cannot have order $c \geq n - 4$, contrary to inequality (29).

If $m \geq 3$ and $G$ has a complete subgraph $K_m$ then $c \leq |V(G/K_m)|$. If follows that in this case we have

$$c \leq |V(G/K_m)| = n - m + 1. \quad (30)$$

By (27), (28) and (30),

$$n \leq p + 3 - m + \frac{4}{m-1} + \frac{m}{4(c-p)} + \frac{1}{4(c-p)} + \frac{1}{4(c-p)(m-1)}. \quad (31)$$

If $m \geq 4$ then by $c \geq p + 1$ and (30),

$$p + 4 = (p + 1) + 4 - 1 \leq c + m - 1 \leq n.$$

From another way, by (31) and $c - p \geq 1$,

$$n \leq p + 3 - m + \frac{3}{4} + \frac{m}{4} + \frac{1}{4} + \frac{1}{12},$$

$$n \leq p + 3 - \frac{3}{4}m + \frac{5}{3},$$

$$n \leq p + 3 - \frac{3}{4}(4) + \frac{5}{3} = p + \frac{5}{3},$$

a contradiction.

If $m = 3$, then by (28) and $c \geq p + 1$, we have $n \geq 3 + p$. Hence $n = p + 3$, and so $c = n - 2$. This shows that $G$ contains a triangle $H = K_3$ such that $G' = G/H$ on $p + 1$ vertices and

$$|E(G')| = |E(G)| - 3.$$

As a special case of (16), we have that

$$t(3, n - p + 1) + 2p - 4 \leq |E(G)| \leq t(3, n - c + 1) + 2c - 4,$$

and so,

$$t(3, 4) + 2(n - 3) - 4 \leq |E(G)| \leq t(3, 3) + 2(n - 2) - 4.$$

Therefore,

$$|E(G)| = 2n - 5.$$

Hence,

$$|E(G')| = |E(G)| - 3 = (2n - 5) - 3 = 2(n - 2) - 4 = 2c - 4.$$
By Theorem G and $c = p + 1$, $G' = K_{2,c-2} = K_{2,p-1}$. By (13), $p = c - 1 \geq 3$ and so (c) of Theorem 1 holds.

**Subcase B2** $c = n$. Then by (13) $n \geq 4$ and $G$ is a reduced graph. By Theorem F(c) $G$ is $K_3$-free. Hence $m = 2$. By (14)

$$|E(G)| \leq 2n - 4. \tag{32}$$

By (31),

$$n \leq 2 + p + 4 + \frac{1}{n - p}. \tag{33}$$

If $n = p + 1$ then by the hypothesis of Case B, $2 = m < n - p + 1 = 2$, a contradiction.

If $n \geq p + 2$. Then by (33),

$$p + 2 \leq n \leq 2 + p + 4 + \frac{1}{2}. \tag{34}$$

$$p + 2 \leq n \leq 6 + p. \tag{35}$$

By (35), (5) and routine computation, we have the following;

$$2n - 4 \geq |E(G)| \geq \begin{cases} 2n - 4 & \text{if } n = 6 + p; \\
2n - 5 & \text{if } n = 5 + p; \\
2n - 6 & \text{if } n = i + p, i \in \{2, 3, 4\}; \end{cases}$$

The conclusion (d) of Theorem 1 holds.

The proof of Theorem 1 is complete. □

4. REFERENCES


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