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# On the edge arboricity of a random graph

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## Abstract

The edge arboricity  $a(G)$  of a graph  $G$  is the minimum number of acyclic subgraphs whose union covers the edge set of  $G$ . In this note we show that if the edge probability is given by  $p^3 n = c \log n$ , then almost every graph has

$$a(G) = \left\lceil \frac{|E(G)|}{n-1} \right\rceil$$

provided the constant  $c$  is sufficiently large.

**Dedicated to Roger Entringer on the occasion of his 60th birthday**

## 1 Introduction

The edge arboricity  $a(G)$  of a graph  $G$  is minimum number of acyclic subgraphs whose union covers the edge set of  $G$ . Nash-Williams [Na64] proved that

$$a(G) = \max_{H \subseteq G} \left\lceil \frac{|E(H)|}{|V(H)| - 1} \right\rceil \quad (1.1)$$

where the maximum runs over all non-trivial induced subgraphs  $H$  of  $G$ . The first two authors showed [CaC91] that when the edge

probability  $p$  is fixed, almost all graphs  $G$  have the property that  $|E(H)|/(|V(H)| - 1)$  attains its maximum in (1.1) if and only if  $G = H$ . Following closely the method of [CaC91], we will extend that result for  $p = p(n) \rightarrow 0$ .

Our sample space consists of all labeled graphs  $G$  with  $n$  vertices. The vertex set of  $G$  is  $V(G) = \{1, 2, \dots, n\}$  and the edge set is  $E(G)$ . Given the edge probability  $0 < p < 1$ , the probability of a graph  $G$  with  $M$  edges is defined by

$$P(G) = p^M(1 - p)^{N-M} \quad (1.2)$$

where  $N = \binom{n}{2}$ , the number of slots available for edges. Thus the sample space consists of Bernoulli trials and the edges are selected independently with probability  $p$ . Suppose  $\mathcal{Q}$  is a set of graphs of order  $n$  with some specified property  $Q$ . If the probability  $P(\mathcal{Q})$  approaches 1 as  $n$  goes to infinity, then we say that *almost all graphs have property  $Q$  or the random graph has property  $Q$  a.s.* (almost surely).

For background material and notation not provided here one can consult the introductory book on random graphs [Pa85] and for the strongest and many of the most recent results we use the extensive and comprehensive treatise [Bo85].

## 2 Edge arboricity

For any non-trivial, connected graph  $G$  of order  $n$ , define

$$\gamma(G) = \max_{H \subseteq G} \frac{|E(H)|}{|V(H)| - 1}, \quad (2.1)$$

where the maximum is taken over all non-trivial subgraphs  $H$  of  $G$ . We use the following elementary inequalities frequently:

$$\frac{|E(G)|}{n-1} \leq \gamma(G) \leq \lceil \gamma(G) \rceil \leq a(G). \quad (2.2)$$

Let  $\mathcal{F}(G)$  be the family of non-trivial subgraphs  $H$  of  $G$  such that

$$\gamma(G) = \frac{|E(H)|}{|V(H)| - 1}. \quad (2.3)$$

Thus these graphs  $H$  achieve the maximum value in (2.1) and it is also easy to see that  $\gamma(H) = \gamma(G)$ .

**Theorem 2.1.** With edge probability defined by  $p^3 n = c \log n$ , if the constant  $c$  is at least 28, almost surely  $\mathcal{F}(G) = \{G\}$  and hence the edge arboricity is

$$a(G) = \left\lceil \frac{|E(G)|}{n-1} \right\rceil. \quad (2.4)$$

Proof: Suppose  $G$  is any connected graph of order  $n > 1$ . Let  $H$  be in the family  $\mathcal{F}(G)$  and set  $r = |V(H)|$ . First we find a lower bound for  $r$  in terms of the number of edges of  $G$ . By the definition of  $H$  we have

$$\gamma(H) = \gamma(G) = \frac{|E(H)|}{r-1}. \quad (2.5)$$

Now we combine (2.2) and (2.5) and use the fact that  $H$  has order  $r$  to obtain .

$$r = \frac{2}{r-1} \binom{r}{2} \geq \frac{2}{r-1} |E(H)| = 2\gamma(G) \geq 2 \frac{|E(G)|}{n-1}. \quad (2.6)$$

Next we can use Chebyshev's inequality to derive an approximation for the number of edges in a random graph  $G$  from which we can determine a lower bound for  $|E(G)|$ . See, for example, a special case in exercise 3.1.2 of [Pa85]. For a slightly more general result, we have the following. For any positive sequence  $\varepsilon_n \rightarrow 0$ ,

$$|E(G)| \geq p \binom{n}{2} (1 - \varepsilon_n), \quad (2.7)$$

provided that  $\varepsilon_n^2 p n^2 \rightarrow \infty$ .

By hypothesis our edge probability is well beyond the threshold for connectedness (see [Bo85] or [Pa85]) so we can assume that almost all graphs are connected.

Combining (2.6) and (2.7) we observe that for almost all graphs, the number  $r$  of vertices in a graph  $H$  from the family  $\mathcal{F}(G)$  satisfies

$$r \geq p n (1 - \varepsilon_n), \quad (2.8)$$

provided that the condition in (2.7) on  $\varepsilon_n$  is satisfied.

At this point we need an estimate for the number of edges in  $H$ . Using Theorem 8, p. 44 of [Bo85], we can conclude that  $|E(H)|$  is almost surely quite close to  $p \binom{r}{2}$ . In particular, we can conclude that

$$\gamma(H) = \frac{|E(H)|}{r-1} \leq \frac{r}{2} \left\{ p + \left( \frac{7p \log n}{r} \right)^{1/2} \right\}, \quad (2.9)$$

almost surely, provided that

$$r \geq (252/p) \log n. \quad (2.10)$$

And this latter condition will be met if the lower bound in (2.8) exceeds the right side of (2.10), i.e. we just need

$$pn(1 - \varepsilon_n) \geq (252/p) \log n. \quad (2.11)$$

On solving this equation for  $p$ , we find that all required conditions on  $p$  are met if  $p$  is defined as in the hypothesis.

Now we are ready to compare  $n$  and  $r$  by using the lower bound on  $\gamma(G)$  in (2.6) and (2.7) and the upper bound on  $\gamma(H)$  from (2.9). Since  $\gamma(H) = \gamma(G)$ , we have

$$\frac{r}{2} \left\{ p + \left( \frac{7p \log n}{r} \right)^{1/2} \right\} \geq \frac{pn}{2} (1 - \varepsilon_n). \quad (2.12)$$

On substituting the expression from the hypothesis for  $p$  in this inequality, after a few steps we find that

$$n - r \leq c_0 (n \log n / p)^{1/2}, \quad (2.13)$$

for suitable  $\varepsilon_n$  and where  $c_0$  is a constant greater than  $\sqrt{7}$ .

Now suppose that there is a vertex  $v$  of  $G$  that is not also in  $H$ . We are going to find an upper bound for the degree of  $v$  in  $G$  that is too far from the mean to hold for almost all graphs. This will imply that such vertices almost surely do not exist. Define  $H_v$  to be the subgraph of  $G$  induced by  $v$  together with the  $r$  vertices of  $H$ . By the definition in (2.1),

$$|E(H_v)| \leq \gamma(H_v)r. \quad (2.14)$$

But since  $H$  achieves the maximum value in (2.1),

$$\gamma(H_v) \leq \gamma(H). \quad (2.15)$$

Combining (2.14), (2.15) and (2.5), we have

$$|E(H_v)| \leq \gamma(H_v)r \leq \gamma(H)r = |E(H)| + \gamma(H). \quad (2.16)$$

This implies that the degree of  $v$  in  $H$  is at most  $\gamma(H)$ , and hence the degree of  $v$  in  $G$  is at most  $n - r + \gamma(H)$ , i.e., almost surely

$$\deg_G v \leq n - r + \gamma(H). \quad (2.17)$$

Using the bounds in (2.9) and (2.13), we find

$$\deg_G v \leq c_0(n \log n/p)^{1/2} + \frac{r}{2} \left\{ p + \left( \frac{7p \log n}{r} \right)^{1/2} \right\}. \quad (2.18)$$

And after a bit of work on the right side of (2.18) in which the value of  $c_0$  depends on  $c > 28$ , we have

$$\deg_G v \leq (1 - \varepsilon)pn, \quad (2.19)$$

for large  $n$  and sufficiently small  $\varepsilon > 0$ .

This last inequality contradicts a theorem of Erdős and Rényi which states that if  $pn/\log n \rightarrow \infty$ , then almost surely the degrees of all vertices satisfy

$$(1 - \varepsilon)pn < \deg_G v < (1 + \varepsilon)pn. \quad (2.20)$$

where  $\varepsilon > 0$  is arbitrary (see p. 66 of [Pa85]). //

We suspect that the theorem gives the right value for the edge arboricity for much lower edge probabilities but the family  $\mathcal{F}(G)$  may not consist of  $G$  alone.

### 3 Tree packing number

The *tree packing number*  $t(G)$  of a connected graph  $G$  is the maximum number of edge-disjoint spanning trees contained in  $G$ . It can be used as a measure of network vulnerability and is closely related to the edge arboricity  $a(G)$ . And the same method of [CaC91] can be applied here with the same result. Tutte [Tu61] and Nash-Williams [Na61] proved that

$$t(G) = \lfloor \eta(G) \rfloor, \quad (3.1)$$

where

$$\eta(G) = \min_{E \subseteq E(G)} \frac{|E|}{c(G - E) - 1} \quad (3.2)$$

and  $c(G - E)$  is the number of components of  $G - E$ .

For any graph satisfying  $\mathcal{F}(G) = \{G\}$ , we always have  $\gamma(G) = \eta(G)$  (see [CaGHL92]). But it can be shown that almost surely  $\gamma(G)$  is not an integer and hence random graphs for which  $\mathcal{F}(G) = \{G\}$ , have

$$a(G) = t(G) + 1. \quad (3.3)$$

Of course, we only have found the values of these packing and covering numbers for random graphs when  $p$  is defined as in Theorem 2.1.

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