SCRABBLE AND RETROGRADE ANALYSIS

SIMON NORTON
Cambridge, England

Reading Mike Keith's article "Scrabble-Tile Double Word Squares and Rectangles" (WW Feb 2011) led
me to ask the following question: which of the word grids he gave are actually legal positions in a game of
Scrabble? Chess problemsists use the term "retrograde analysis" for the study of how chess positions might
have arisen in a legal game; this concept is of course equally applicable to other games, including Scrabble.

I should say, for the benefit of readers who may wish to pursue any of the questions I ask, that when
writing this article I have not attempted to consult any online word lists; all the words used here are either
words I know or words used in Mike Keith's article.

The rules of Scrabble are easy to axiomatise. (The following includes references to fixed numbers - 7,
15, 100 - which are correct for the standard game of Scrabble, but which can in principle be changed.)

1. The alphabet is a set of symbols called letters. A word is a sequence of at least 2 letters. (Note
that 1-letter words play no role in Scrabble.) A subword of a word is a sequence of at least 2 letters
obtainable by removing (possibly zero) letters from the beginning and end; it is proper if at least one letter
is removed. The lexicon is a set of words (those which are allowed in Scrabble). The ensemble is the
collection of letters, each with a given multiplicity, which can be used in a Scrabble position. Apart from a
letter for each lettered tile, the ensemble includes "wild cards" - the equivalent of the blank tiles which can
represent any letters - and its total size is 100.

2. A polyomino is a geometrical figure, made up of unit squares, where any two squares are connected
by a sequence in such a way that adjacent squares in the sequence share a common (full length) edge. The
board is a particular polyomino, a square of side 15, whose central square plays a distinguished role. A
polyomino is possible if it has between 2 and 100 squares and can be placed within the board so as to include
the central square. In a polyomino a chain of unit squares consists of an $n \times 1$ or $1 \times n$ rectangle (with
$n \geq 2$) entirely within the polyomino, and such that if extended (in either of the two possible directions) to
an $n+1 \times 1$ or $1\times n+1$ rectangle then it is no longer entirely within the polyomino.

3. A crossword is a polyomino, with left-right or up-down directions defined on every chain in a
compatible way, in which each unit square is labelled with a letter. It is possible if the polyomino is
possible, and if every chain, read left to right or top to bottom, forms a word within the lexicon. A word is
possible if its letters fit within the ensemble (i.e. excluding words like RAZZMATAZZ), and if it fits within
the board (i.e. has at most 15 letters).

4. An initial move consists of the setting up of a chain, of length at most 7, which is labelled and
oriented to form a word within the lexicon and positioned on the board in such a way as to include the
central square. A move is either an initial move, or an operation that takes a crossword $A$ to a crossword $B$,
both possible and with the same orientation, by inserting between 1 and 7 squares, labelled with letters,
in such a way that there is a single word of $B$ which contains all the inserted letters. An unmove is the
reverse of a move.

5. A crossword is legal if it is the result of an initial move followed by a (possibly empty) sequence
of moves. A polyomino is legal if it can be labelled with letters so as to form a legal crossword. A word
is legal if there is a legal crossword in which it appears as a word. A polyomino is true if any crossword
obtained by labelling it is legal if it is possible, and if this remains so with an arbitrary lexicon. (That means
essentially that it can be built up without using any proper subwords of the words that appear in it.) A
grid is a rectangular polyomino in which each dimension is at least 2. Without loss of generality we can
assume that its length - the distance from left to right - is at least as great as its height - the distance from
top to bottom. (This was the case with all the grids depicted in Mike Keith's article.)

Note the following:

(a) I am assuming that all 100 tiles can be placed on the board. This is not true of a real game with at
least 2 players, as a game ends as soon as any player has no tiles left. In general, in an $n$-player game, the
maximum number of tiles that can be placed before the game ends is $101 - n$. But in a solitaire game all
100 tiles can be placed (as the above formula implies).
Q1: Is there any other way in which a crossword can be legal in a solitaire game but not in a game with at least 2 players? To show how this might happen, imagine a 2-player game in which 86 tiles had been placed. This would leave each player with 7 tiles. It would therefore be impossible to extend this game by three 4-letter moves. Can one find a crossword which cannot be reached by a different sequence of moves?

(b) The above does not involve itself with matters of scoring, or the possibility of “pass moves” in which a player exchanges letters. Neither is relevant to the question of which positions can be the result of a legal game.

To start the ball rolling (and illustrate the definition of trueness) we prove that any grid of size at most $8 \times 3$ is true. In fact, we can state a stronger result: a crossword based on such a grid can be positioned with any letter being at the central square of the board, and however this is done the crossword will be legal. The proof is simple: choose as the initial move the column (of length 2 or 3) containing the central square; then complete the top row; then complete the bottom row; then, if the height of the grid is 3, complete the middle row. It can easily be seen that the only words created are those that appear in the full grid.

This means that the smallest possible grid which might be possible but not legal is $4 \times 4$. It is easy to see that such a grid is not true, because any possible unmove must create at least one new word (in fact more than one).

Q2: Is there a $4 \times 4$ grid which is possible but not legal? If not, what is the smallest grid of this type?

Mike Keith’s article included the “heterogrammatic” $4 \times 4$ grid shown below (Table 1, centre). This grid is legal: one can start by removing the B, then the S, E and W, then the rest is easy.

There are polyominos smaller than the $4 \times 4$ grid which are not true; this can happen when there are words too long to be playable in a single move, but it can also happen for the same reason as with a $4 \times 4$ grid. I am pretty sure that the smallest case of the latter is a $4 \times 4$ square with its corners removed.

Q3: Is there a crossword based on this polyomino which is possible but not legal? If this is to happen, none of the proper subwords of its 4-letter words may appear in the lexicon. If there is no crossword of this type, what is the smallest crossword consisting of more than one word that is possible but not legal?

Mike Keith said that there was a unique $7 \times 7$ grid which was possible (and where no two words were the same). This grid is reproduced below (Table 1, left). I was delighted to find that it is legal: it can be created by the following sequence of moves:

A: Start with the word SNEEZER.
B: In any order, lay the words SNIP, EIDOLIC, ZONATED and RESCECTS in the first, third, fifth and bottom rows.
C: In any order, lay the words NOISOME, PHORATE, MI and RECEDES in the second, fourth, sixth and last columns.
D: Complete the words ESERINE, EMINENT, POLITIC and INDENES in that order.

However, I was unable to find a legal game leading to his “high scoring” $7 \times 5$ grid (Table 1, right).

Let me conclude with some further questions for readers:

Q4: Find a polyomino that is possible but not legal. Mike Keith’s article would have proved that this was true of a grid of size larger than those he constructed but for his requirement that all words within any of his grids be distinct. However I am pretty sure that such a polyomino exists.

Q5: Is every word legal if it is possible? Note that even a 15-letter word can be built up independently of whether appropriate subwords are in the lexicon; the relevant unmove is to remove the letters in the 7 even positions and leave the letters in the 8 odd positions. However it is not obvious, though it seems likely, that a crossword can always be chosen in such a way that this unmove is to a possible position and can be followed by further unmoves which prove the crossword legal.