GAMES ON WORD CONFIGURATIONS

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The rules for our proposed games are very simple. Two persons alternately draw letter tiles until one of them is first able to form a word and wins. To avoid arguments, we supply for each game its own dictionary of allowed words. To make the games more interesting, we have certain symmetry demands on dictionary word lists and will call such lists configurations.

By a word configuration we mean explicitly a collection of distinct letters and a collection of distinct words formed from them subject to the following requirements:

R1: any two letters are in at most one word, and any two words have at most one letter in common
R2: each word has r letters, and each letter is in r words
R3: there are the same number, n, of words as letters

We call (n,r) the class of the configuration.

If we had been more stringent and in R1 replaced "at most" with "exactly" we would have the letter-word equivalent of the point-line projective geometry of mathematics. Finding such word sets is not easy (see "The Thirteen Words" May 1972, "The Twenty-One Words" November 1975, and "Word Groups" May 1977). If, on the other hand, we had not insisted in R2 that each letter be in r words, we would have generated a configuration originally exploited by Dudeney (see "Dudeney's Lost Word Puzzle" August 1986). Luckily, it is a bit more straightforward to construct the word configurations of this article.

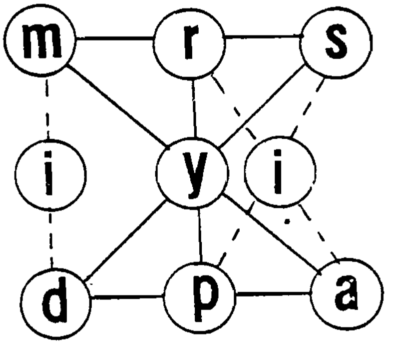
We introduce our ideas with the pleasant little game of Pyramids. The letter list consists of the letters of PYRAMIDS (for aesthetic appeal we like to have our letter list transposable into a word), and the word list is AIR, DIM, MRS, PAD, PRY, SIP, SYD and YAM. The reader will be able to verify that this is a word configuration of class (8,3). The game can be played by drawing blindly from PYRAMIDS, but we prefer to allow the players to be able to choose the tiles they want. With this proviso, Pyramids is a first-person win, and we invite the reader to experiment and try to find the winning strategy.

To add excitement to Pyramids, we sometimes insist that the first player (hereafter A) is to roll a special die for his first choice. The die is made from a rectangular octahedron on which the eight letters are printed, one to a face. (It is refreshing to note that the octahedron can be made by gluing two pyramids together.) Even if A's first move is decided by roll, he can still force a win as long as he can choose later moves. The second player (hereafter Z), when it is his turn to go first, could, of course, win as easily as A, but if he does not know the strategy behind Pyramids he will find that he only has odds of 1:5 of winning. Templates for the die and playing pieces for Pyramids are on the next page.

Before giving the winning strategy for Pyramids, we introduce the concept of the dual game. Instead of drawing letters, A and Z alternately draw word tiles until one of them corners all the words with a particular letter in them and wins. For example, if A obtains the words AIR, DIM and SIP in his first three plays, he has captured I and wins at dual Pyramids. If we have a game of class (n,r), our three requirements, all having a dual nature, dictate that a dual game also of class (n,r) with "words" and "letters" interchanged. Any strategies we develop for playing a game of class (n,r) will carry over, mutatis mutandis, to the dual one.

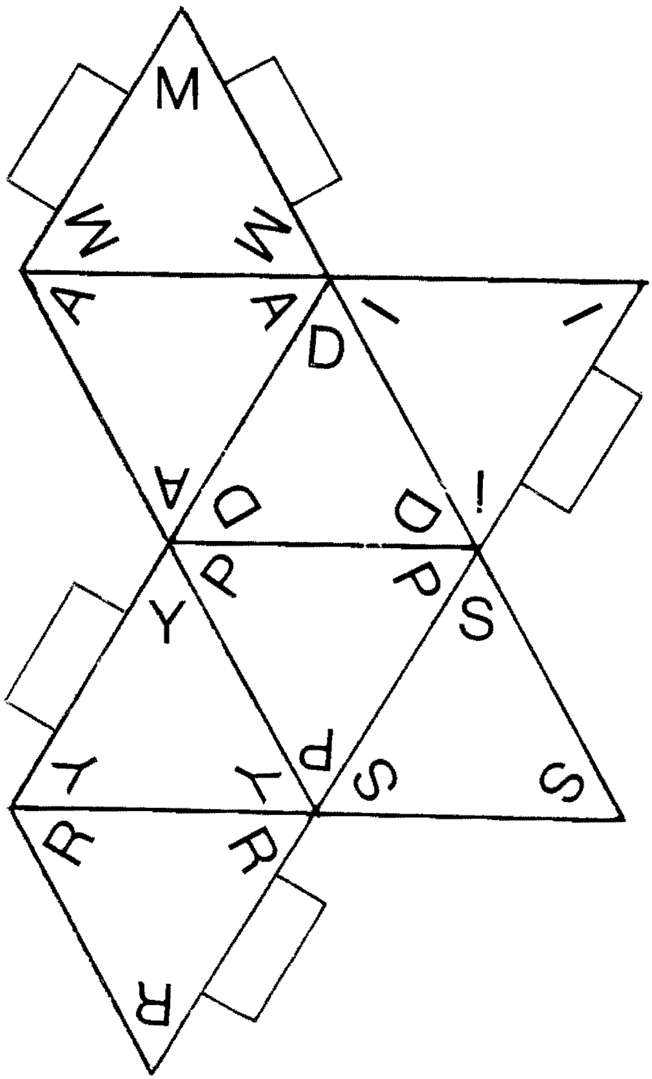
Here is how to win at Pyramids. Memorize the phrase DR AS MP, YI. If you forget this phrase, you can find it again by looking at opposite faces of the octahedral die. Let A take any letter, say P. If Z takes its mate in the phrase, here M, A can take any other letter and win. For example, in order of play: P, M, S (threatens SIP), I (threatens DIM), D (threatens SYD or PAD). If Z plays other than M, A takes the mate of Z's move and wins, e.g. P, R (D is the mate), D (threatens PAD), A (threatens AIR), I (threatens MID or SIP).

Pyramids is similar to other "three-in-a-row" games like tic-tac-toe, but while the latter has a well-known playing board, no board is possible for Pyramids--at least none with points and straight lines. The closest board for Pyramids is given below.



Notice that I is split in the figure, and would have to be doubly occupied when chosen in a board game. This inelegance is not present in the word game version.

There are no other games of class (8,3). Any other proposed (8,3) game would be Pyramids with merely new labels; a mathematician would call the two isomorphic. No other game is possible because there is only one set of "misses" possible. The set of



P	AIR
Y	DIM
R	MRS
A	PAD
M	PRY
I	SIP
D	SYD
S	YAM

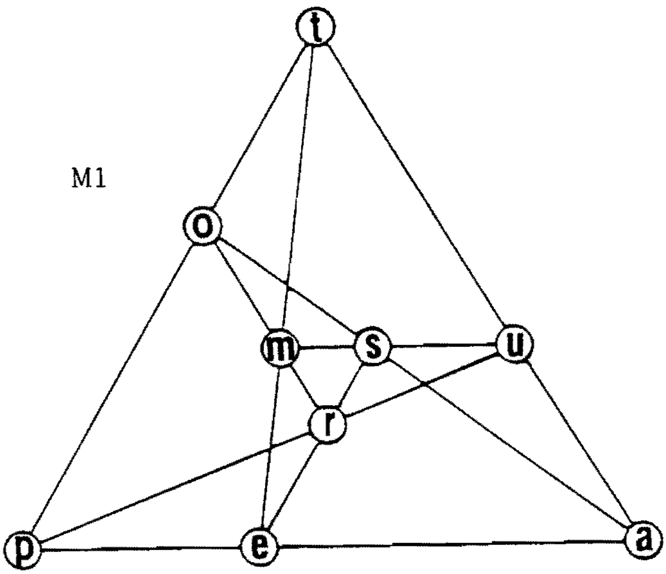
misses is the graph of letters that do not appear together in a word. For Pyramids, this "misgraph" is D-R, A-S, M-P, Y-I, or cycles of length 2,2,2,2. The misgraph for a game of class (n,r) will characterize that game and any other game with the same misgraph will not be regarded as different.

There are exactly three games of class (9,3) which we label as types M1, M2 and M3. All three can be played on the letters of MOUSETRAP and have word lists:

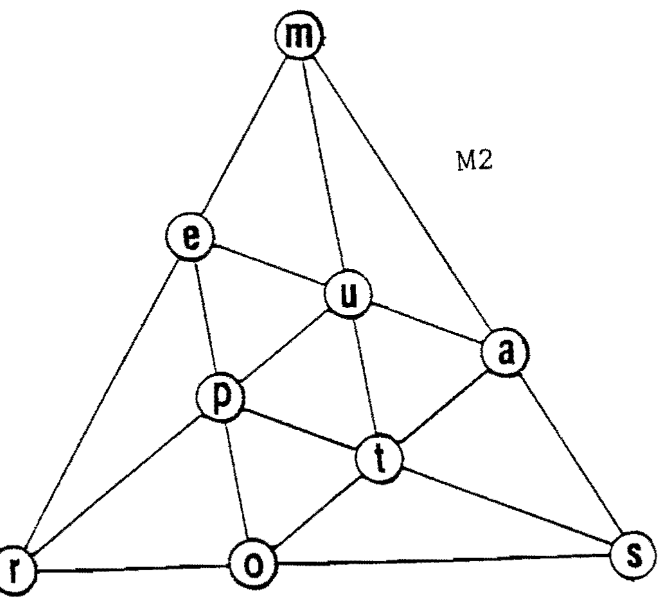
M1: POT, MET, TAU, ROM, OSA, SUM, ERS, PRU and APE
M2: SAM, TUM, MER, POE, EAU, ORS, PTS, OAT and PRU
M3: MAP, OUR, SET, MOE, SUP, PER, ART, TUM and OSA

Each of these games has a geometric playing board, given below.

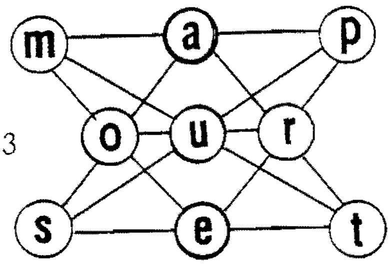
M1



M2



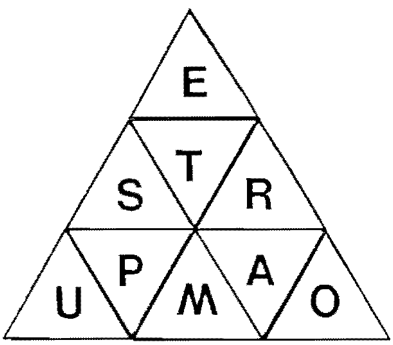
M3



A has a forced win in these three games also, and it can be found on the spot if playing on a board, but the winning strategies are much harder to find when they are played purely as word games.

To win at M1, A should choose one of the letters E, U or O. If Z also selects one of these three, A should take the third and he will win, e.g. U, E, O, S (threatens ERS), R (threatens PRU or ROM). If Z does not choose one of E, U or O, he will take a letter from TRAMPS. If the letter Z chooses is under the letter A has previously chosen, A should next play either letter that is one step away in TRAMPS; if the letter Z chooses is not under the letter A has previously chosen, A should next play the letter three steps away in TRAMPS. The first game is illustrated by U, P, M (or S) (threatens SUM), S (no threat), T (threatens MET or TAU); the second, by U, M, T (threatens TAU), A (no threat), P (threatens POT or PRU). This rather complicated four-move win for A can be simplified by displaying the mystic triangle-hexagon below when playing. We tell our opponents that it has arcane powers that force wins for us. A bit of study will show the truth of this! When you must play second, and your opponent starts with a letter from TRAMPS, it will be from one of the words PAT or MRS. On your play, choose from the same word and this will assure you of at least a draw.

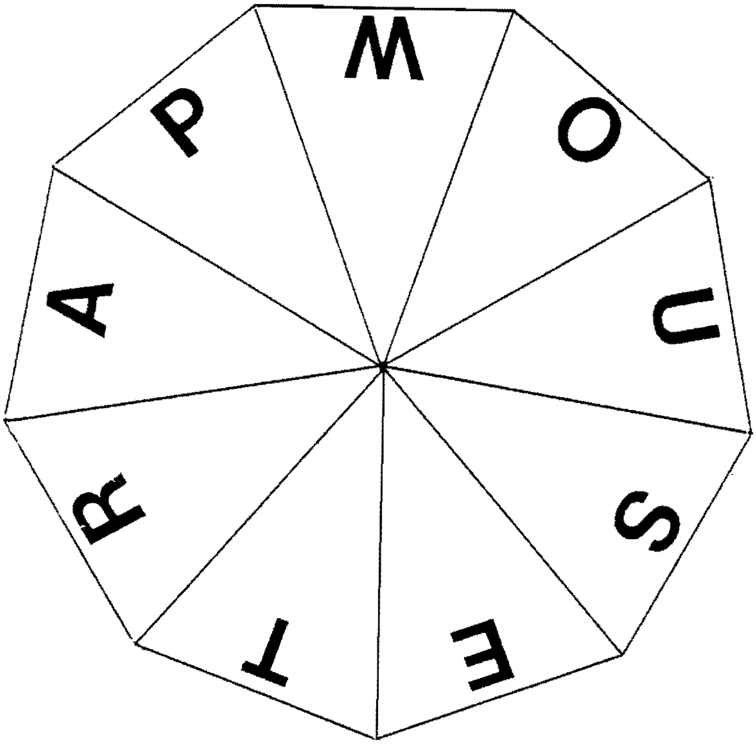
E	U	O	E	U	O
T	R	A	M	P	S



In **Mathematical Magic Show** (Knopf, 1977) on page 69, Martin Gardner gives the geometric version of M1, called Tri-Hex by its inventor, T.H. O'Beirne. Even with the board diagram the game is difficult.

We have designed M2 around the nine-sided polygon with the consecutive letters of MOUSETRAP in cyclic order around the sides (see the figure on the next page). To win, A can take any letter, say M, and imagine the enneagon rotated so that M is at the top. One would obtain the letters TRAPMOUSE if laid out straight. A should memorize the scheme diagrammed at the left in which the arrows mean to either TAKE the letter pointed to, or force that letter, for any choice by Z. To illustrate, M, A, R (threatens MER), E (threatens EAU), U (threatens PRU or TUM); or M, P, S (threatens SAM and therefore forces A), A (no threat), R (threatens MER or ORS). It is not hard to remember the scheme if an old spinner from a child's game is revamped by placing the enneagon on it and A spinning for his first move. There is one move

→	←	← TAKE	←		→	TAKE →	→	←
T	R	A	P	M	O	U	S	E



by Z (here U) that can cause A's win to take five moves. This defect is rare but cannot be avoided.

M	R	S
E	A	U
P	O	T

For M3, A memorizes the 3x3 grid at the left. A chooses any letter, say M. If Z chooses in the same row of the grid, A takes the remaining letter in the row and wins by playing rationally, e.g., M, S, R, O (threatens OSA), A (threatens MAP or ART). If Z chooses from a different row, then A's letter and Z's letter are part of a legal word. A finds the third letter of that word and chooses either of its mates in its grid row, e.g., M, O (the word is MOE), A (or U) (threatens MAP), P, T (threatens TUM or ART). When you play second, we suggest taking in the same grid row as your opponent's choice; he then has only one winning response.

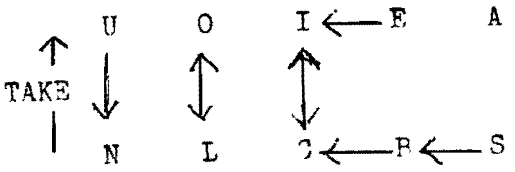
The misgraphs for M1, M2 and M3 are respectively cycles of size 3-6, 9, and 3-3-3-3. Our game Pygmalion (see the November 1992 **Word Ways**) is isomorphic to M3 since it has a misgraph of 3-3-3 as the reader may verify. Exercise: verify that the list TAU, ERS, MOP, ROT, SUM, APE, MET, OSA, PRU is a word configuration on MOUSETRAP of class (9,3) and determine its type.

The only other possibility for a misgraph of class (9,3) would have a 4-5 cycle, but experimentation shows that this does not lead to a word configuration. Notice how each winning strategy is ultimately based on that game's misgraph.

Things get more complicated as we move to larger n. We call class (10,3) the UNSOCIABLE games. We do not know how many types there are of this class but we have found three and will describe two of them as types U1 and U2. On UNSOCIABLE the word lists are:

U1: AIL, SUI, BIN, BOA, BUL, CAN, COS, ONE, CUE and LES
U2: BOA, LIE, SUN, CAN, CUE, COL, SIB, OUI, BEN and SAL

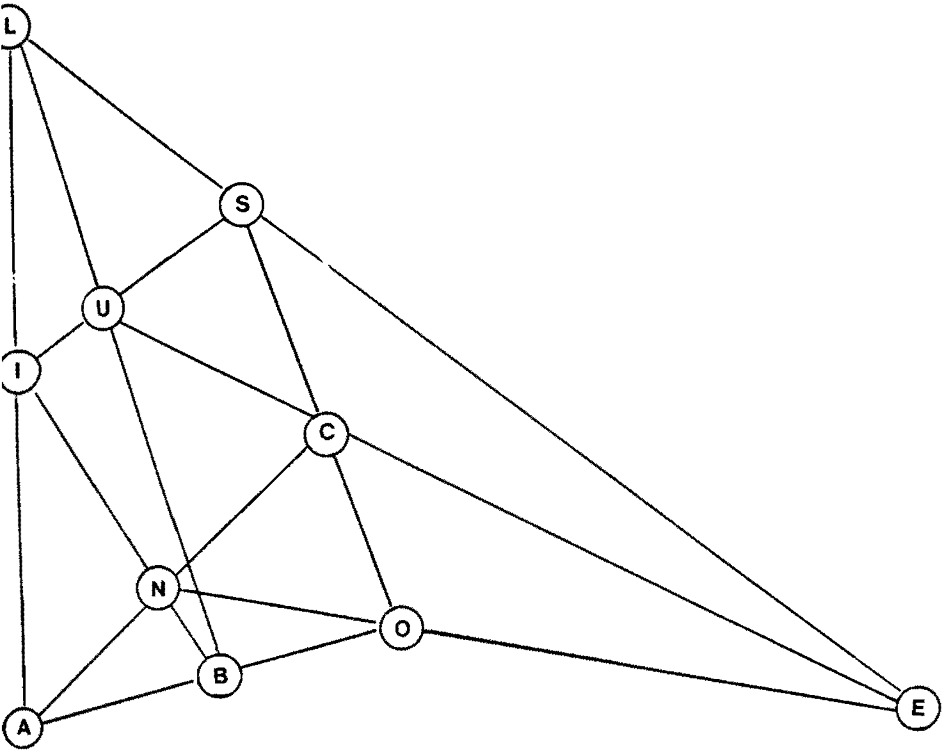
The boards for U1 and U2, as well as their misgraphs, are given on the next pages. The misgraph for U1 is to be regarded as continuing indefinitely, or equivalently it may be viewed as a bracelet with the five vowels over the five consonants. The misgraph of U2 is drawn on a three-dimensional prism. A again has forced wins in both games. For U1, A takes any letter, say A. Turn the bracelet so that A is at the upper right. If the bracelet were to be cut and laid out flat, it would look as shown at the left. A has taken letter A, and whatever letter Z chooses, A must either force the letter in the direction of the arrow or TAKE the letter and then play rationally. For example, A, C (want to force I), L (threatens AIL), I (no threat), B (threatens BUL or BOA); or A, N (want to take U), U (no threat), O (threatens ONE), E (threatens CUE), C (threatens COS), S (threatens LES or AIL).

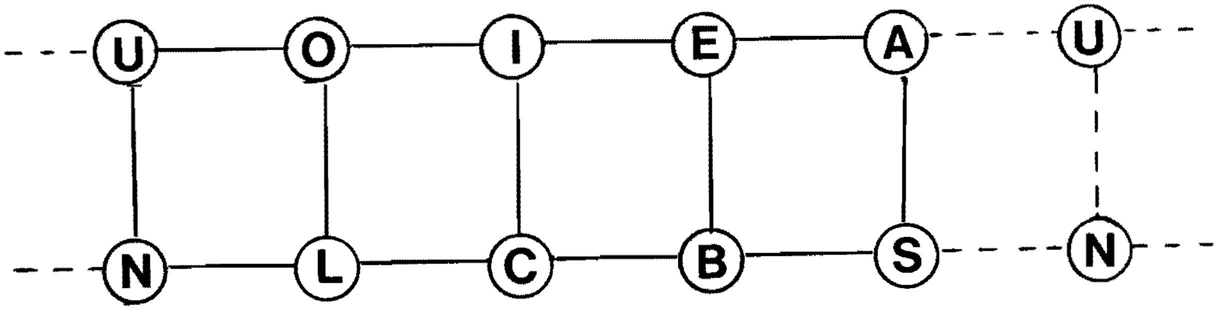


U2, like PYRAMIDS, is defective as a board game since C has to be split. A can win from any position, but the win is easiest from I (the details are left for the reader to find). Unfortunately, this would quickly be discovered by our opponent, so it would be better for A to choose from the letters BUL (or symmetrically OSE). If Z also takes from BUL, A should take the third member, e.g., B, U, L, C (threatens CUE), E (threatens BEN or LIE). If Z takes from OSE, A forces another from OSE, e.g., B, O, I (threatens SIB), S, E (threatens BEN or LIE). If Z takes I, then A forces two from CAN and subsequently wins, e.g., B, I, E (threatens BEN), N, O (threatens BOA), A, C (threatens CUE or COL). If Z chooses from CAN, A wins by forcing a letter from OSE opposite Z's choice.

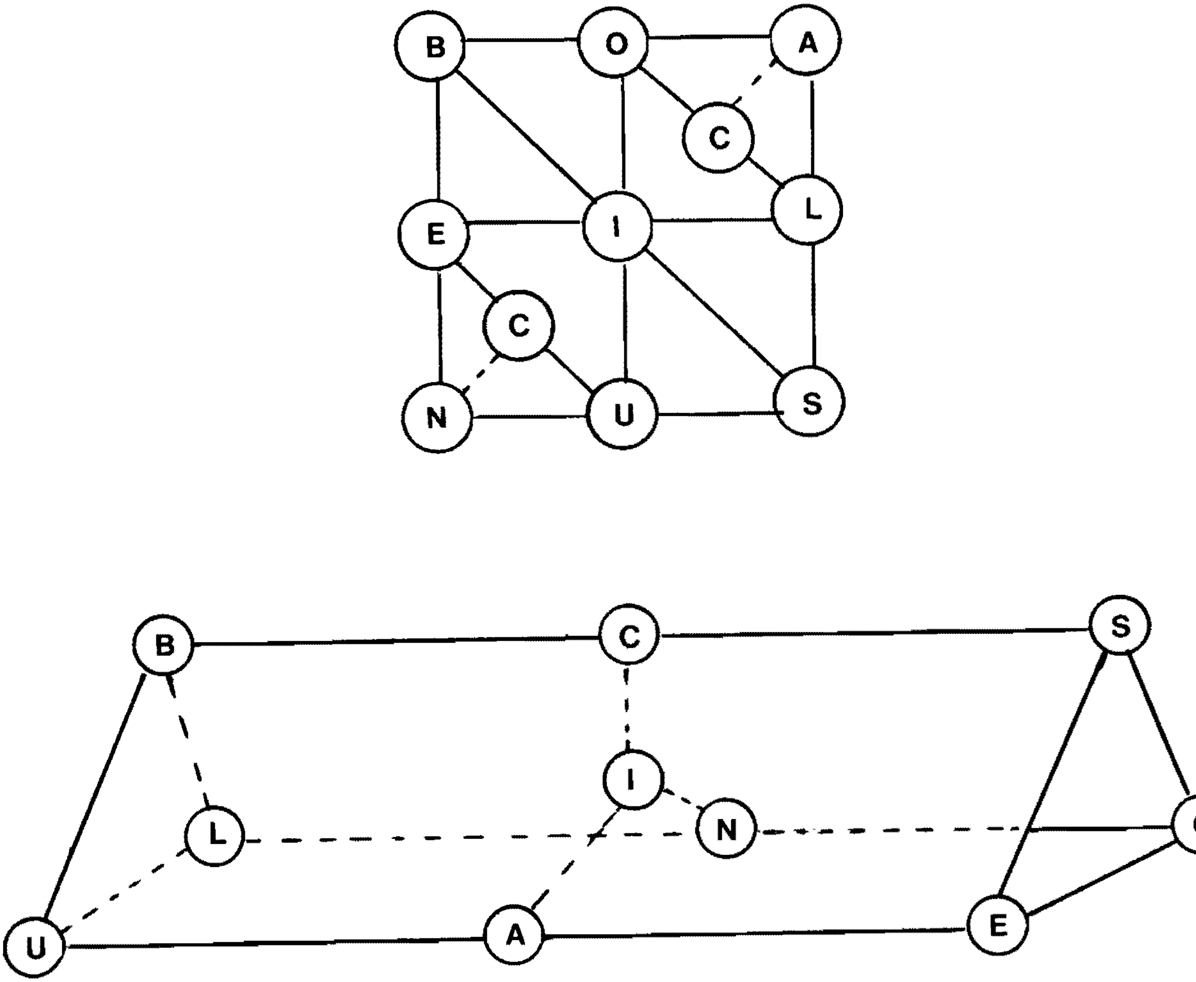
If the reader follows along on the game boards, defective or not, and keeps the misgraphs in front of him, he will see that the wins are not overly complicated. Nevertheless, it takes a real expert to keep all the nuances in mind when playing a variety of configurations, and it is probably best to dwell on just a few. Also, the reader can construct the dual games for any configuration (a list for P, M1, M2, and M3 are in Answers and Solutions) and determine the winning strategies for each.

The Mousetrap games (9,3) have an especially appealing nature, and we suggest a generalization: find other games of class (n,r) where n is the square of r. For (4,2), the four letters can be regarded as the rows and the four words as the columns of a 4x4 grid of 0s and 1s with exactly two 1s per row and two 1s per column. The reader should convince himself that the grid on page 204 (left) is the only one possible, keeping in mind that re-





Game board and misgraph for Unsociable 1.



Game board and misgraph for Unsociable 2.

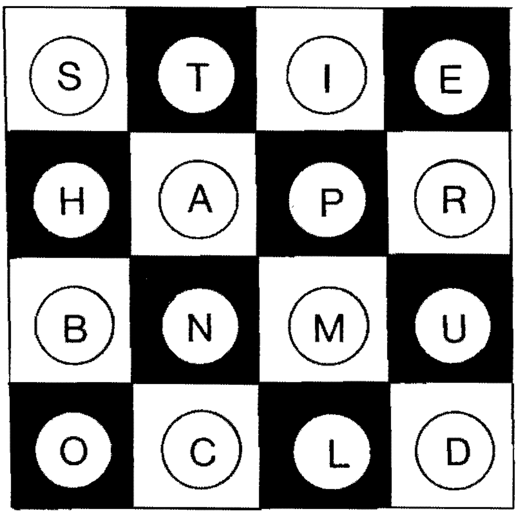
$$\begin{matrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{matrix}$$

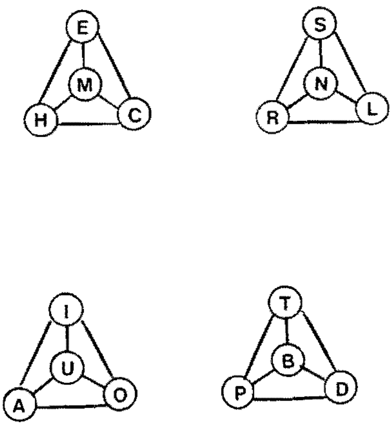
arranging the rows or columns does not count as something different.

So, one word game is possible, and the problem is now to find a four-letter word that breaks appropriately into four two-letter words. SAID works, and can be used as the row headings to obtain the column words AS, IS, AD and ID. The winning strategy is obvious.

We have made some progress on a wonderful class (16,4) game. Its working title is SHRIMP BOAT UNCLE D (a vessel named after a favorite relative?) and its word list is BALE, BOSH, CANT, COLD, CRIB, CUSP, DAMS, HARP, HIND, LIMP. LUTH (from the OED, the "Leather Turtle"), MORT, NUMB, OPEN, RUDE, and SITE.

The gameboard and misgraph for the (16,4) game are given below. The chessboard motif is entirely appropriate since the 16 words are found by four rook sweeps on the rows, four rook sweeps on the columns, two bishop moves on black diagonals (ONPE, HTLU), two bishop moves on white diagonals (SAMD, IRBC), and four knight tours (SPCU, EALB, DNIH, OMTR). In addition, the board is spanned by another set of knight tours, one from each of the four tetrahedrons of the misgraph.





The game is best played by placing checkers or coins over the chosen letters on the board while jotting down the choices on a pad. We conjecture that A has a forced win in this game, but our strategy needs to be checked since the game is rather complex.

To win, A chooses any letter and Z will chose in A's misgraph tetrahedron or not. If not Z's choice will be on black or white in some other misgraph tetrahedron; A should choose the remaining letter in that tetrahedron of the same color. This rule should be applied on Z's second move as well, unless it would force A to take the same color for the third time. In that case, A should take an opposite color in Z's second tetrahedron. A then continues by playing rationally.

A sample game: H, L, N (by rule), M, C (by rule), P (threatens LIMP), I (threatens HIND), D, B (threatens CRIB), R, S (threatens BOSH), O (threatens MORT), T (threatens SITE and CANT). Another: H, L, N (by rule), P, B (by rule), M (threatens LIMP), I (threatens HIND), D, S (threatens BOSH), O (threatens COLD), C (threatens CRIB), R (threatens MORT), T (threatens SITE and CANT).

When Z takes in A's misgraph tetrahedron, A should play to maintain a color balance; otherwise apply the former rule.

A sample game: S, N, L (to assure balance), M, C (by rule), B (threatens NUMB), U (threatens CUSP), P, T (threatens LUTH), H, E (threatens SITE), I (threatens HIND), D (threatens COLD and RUDE). Another: S, R (same color, no threat), T, N, L (by rule), B, U (threatens LUTH), H, P (threatens CUSP), C (threatens CRIB), I (threatens LIMP and SITE). Exercise: Find A's forced win after S, R, T, A, B, D.

There is a class (25,5) of configurations with similar chessboard symmetry. It has ten words from rook sweeps, ten from bishop-like moves and five from knight tours. The misgraph consists of five disjoint pentagons with all five vertices connected. We suggest placing five vowels in one pentagon to assure that each of the 25 words contains a vowel.

The smallest value of n that allows a particular r is of some interest. For r=2, n=3 is the smallest; for r=3, n=7; and for r=4, n=13. Examples of these word configurations are given in the earlier-cited **Word Ways** articles. We do not know how many configurations exist of class (n,r), nor do we know if the first player always has a forced win (he will certainly have at least a draw). It would be interesting to find the other (10,3) games; can they all be worked on UNSOCIABLE?

These games are reminiscent of the problem of planting trees in an orchard, proposed by the mathematician J.J. Sylvester about 1868 (W.W.R. Ball, **Mathematical Recreations and Problems**, MacMillan 1892, page 42). One of his orchard problems was to plant nine trees in ten rows, each containing three trees. An answer given by Ball (due to R.C. Abbott) is equivalent to taking our M3 game and adding the row EAU. Doing this destroys the symmetry and it is no longer a word configuration. Nevertheless it can be played as a game; A has an easy win by taking two successive plays from EAU. Interested readers can explore other orchard problems for possible word games.

Finally, it may be possible to find word games among the point-line projections of higher-dimensional polytopes. Some of these skeletons are beautifully symmetric, a key element in disguising strategies in word games. A good source of examples is H.S.M. Coxeter's **Regular Polytopes** (Dover, 1973).