SCRUBWOMAN EDITH MEETS W.R. HAMILTON

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At an Omnium Gatherum in honor of Martin Gardner held in Atlanta in January 1998, one of us presented the following word puzzle for the amusement of the attendees:

A SCRUB TILE PUZZLE

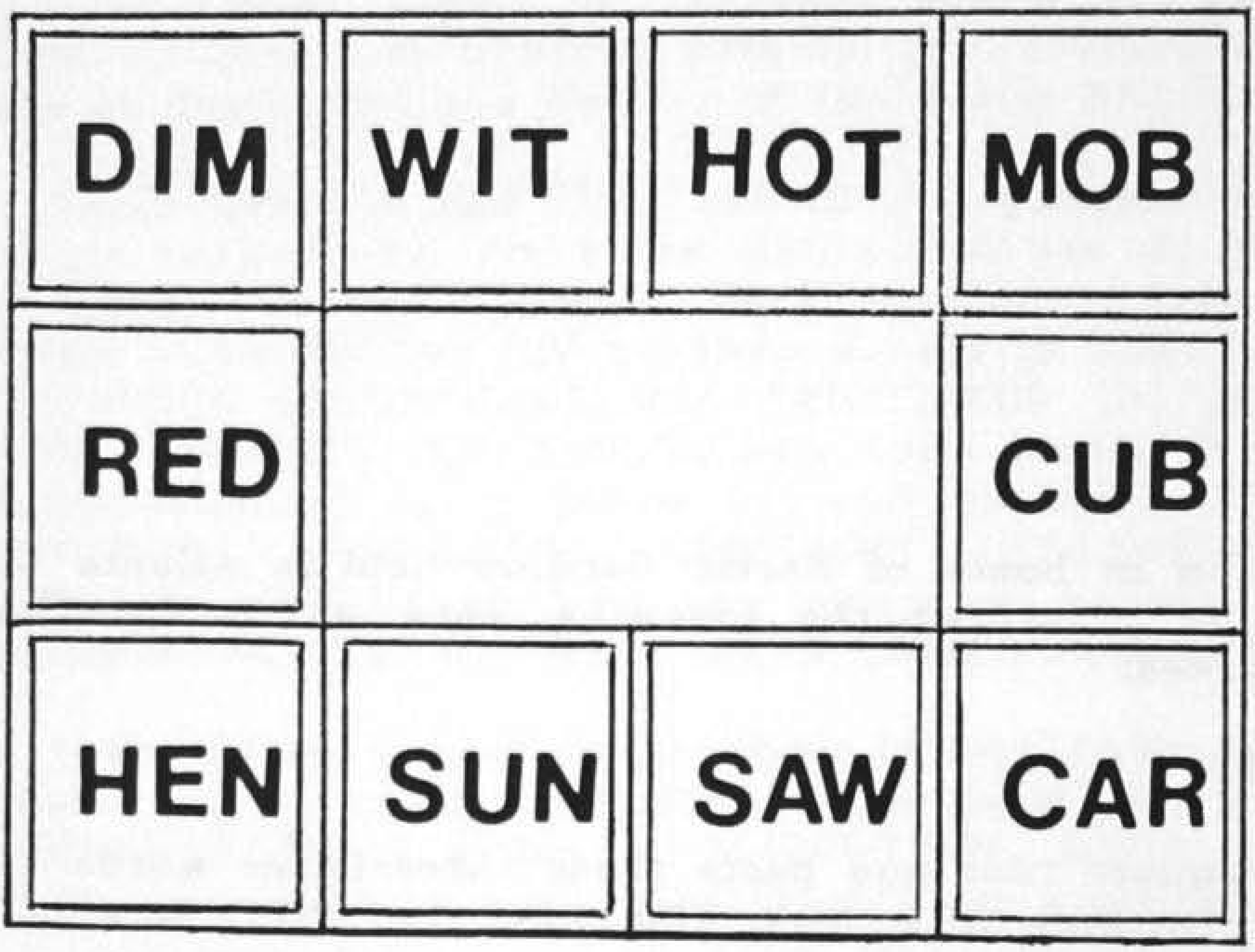
Obtain ten one-inch-square tiles and paste these three-letter words on them: CAR, CUB, DIM, HEN, HUT, MOB, RED, SAW, SON, WIT. Note that every letter occurs exactly twice and that any two letters occur in at most one word. These tiles are called scrubs and operate like word dominoes; that is, two scrubs can abut only if they have a letter in common. It is easy to form a long scrub chain of length ten--this is called a Hamiltonian path--but it is not so easy to make the path loop around and join itself. Can you find such a loop?

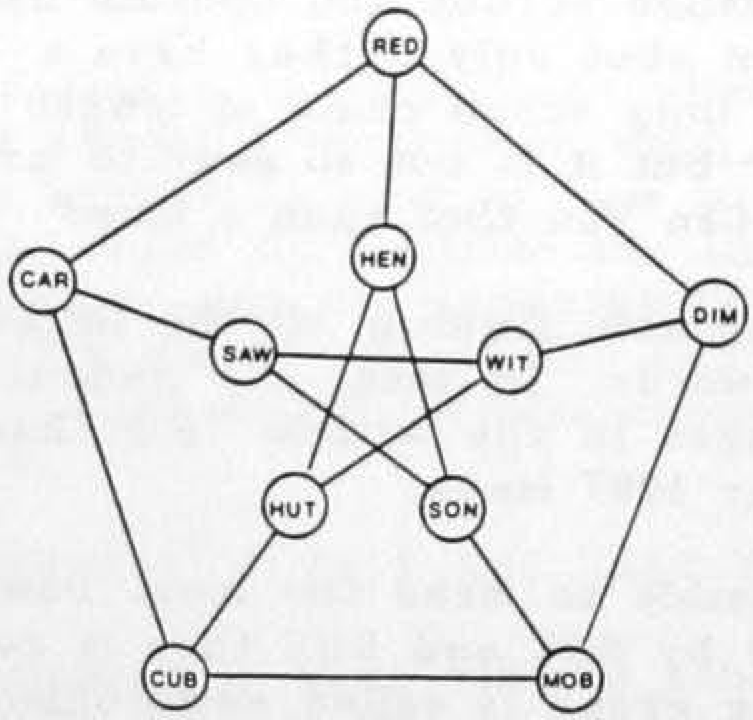
Readers of Word Ways will have no trouble finding all the letters of SCRUBWOMAN EDITH used in the ten words. In fact, the redoubtable Edith was first introduced on these pages in the article "F.P. Ramsey, Meet Scrubwoman Edith" in the November 1997 issue.

But for this puzzle, it is quite impossible to make the loop. However, if the scrubs HUT and SON are replaced by HOT and SUN then it can be done. When it is possible, the underlying graph is called Hamiltonian.

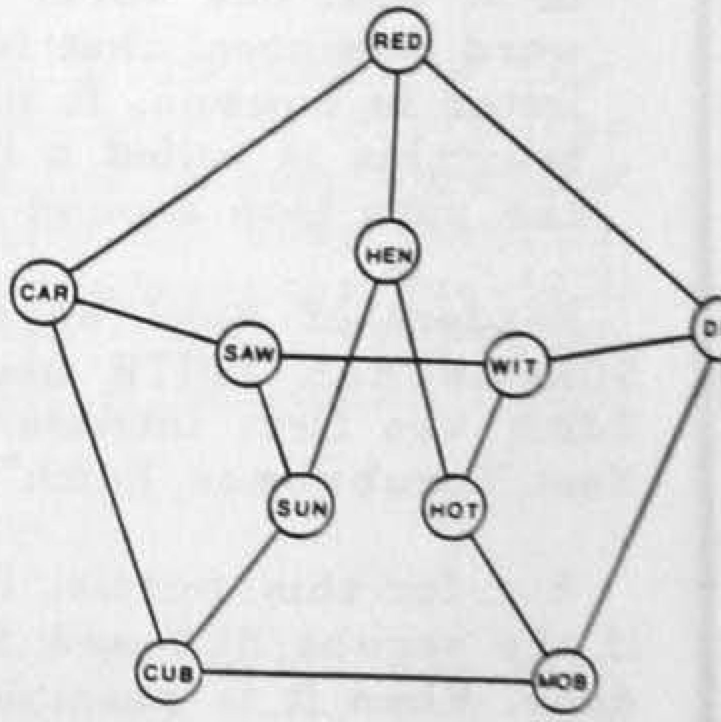
Make a copy of the diagram on the next page and paste it on heavy cardboard. Find some likely victim and arrange your scrubs on the grid shown, palming the HUT and SON scrubs. Briefly show your mark the arrangement and then drop the scrubs on the table. Be careful to hold onto the HOT and SUN scrubs with your thumbs, letting HUT and SON fall in their place. Your poor victim will have no chance at a loop and will probably not know why!

Graphs of each version of the puzzle are given on the next page. One is the famous Petersen graph (Julius Petersen, 1898) that is known to be non-Hamiltonian while the other is Hamiltonian. It is relatively easy to find complete loops on the latter.





Petersen Graph



Hamiltonian Graph

D.A. Holton and J. Sheehan, in their book *The Petersen Graph* (Cambridge University Press, 1993) describe another problem amenable to scrub tile analysis.

SOUSSELIER'S PROBLEM (René Sousselier, 1963)

It appears there was a club and the president descided that it would be nice to hold a dinner for all the members. In order not to give any one member prominence, the president felt they should be seated at a round table. It seems that the club was not an amicable little group as each member only had a few friends within the membership and positively detested all the rest. Hence the president thought it necessary to make sure that each member had a friend sitting on either side of him at dinner. Unfortunately, try as he might, he could come up with no such arrangement, and in desperation turned to an astute friend for help. After a time, the friend replied "It's absolutely impossible! However, if any one member of the group can be persuaded not to turn up, then everyone else can be seated next to a friend. And by the way, if you had fewer members in the club you would not be faced with this strange combination of properties."

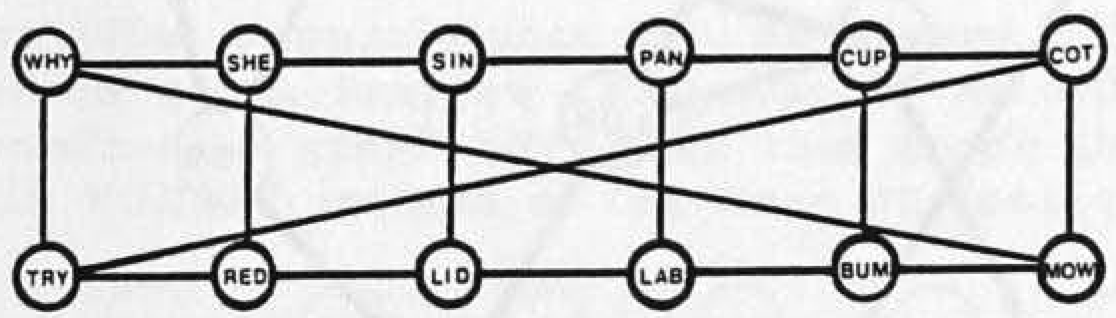
The reader has probably guessed that the astute friend was Edith and the Petersen graph above, regarded as a graph of friendships, solves Sousselier's problem. Holton and Sheehan define graphs that are non-Hamiltonian but upon deletion of any single node (along with all its adjacent edges) do become Hamiltonian as hypohamiltonian.

EDITH'S PROBLEM

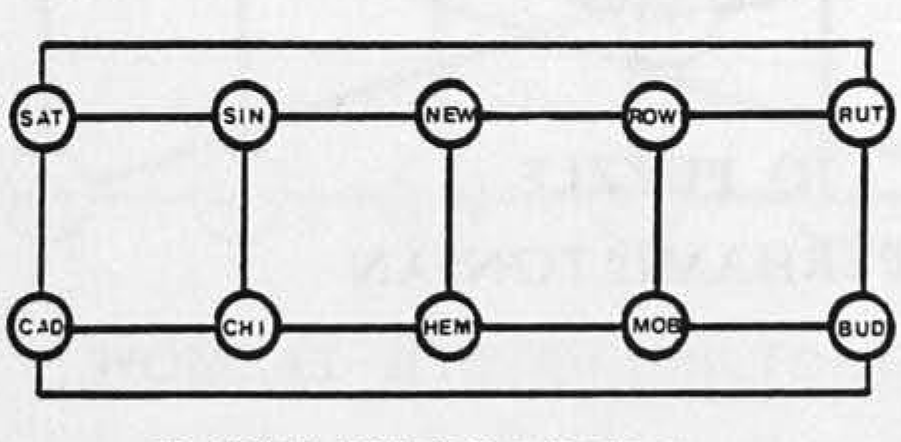
Six married couples, the Andersons, Enders, Ivers, O'Neils, Ungers, and Youngs, are to be seated at dinner around a circular table. It happens that each guest, in addition to his or her spouse, is friends with exactly two other guests. As usual, we want to seat the twelve guests next to people they know. In addition, we are informed that one spouse may not be able to attend but we do not know who it is. Give an arrangement, if possible, of friends that solves this seating problem completely; that is, one that is Hamiltonian on 12 nodes and such that upon deleting any single node can still be made Hamiltonian.

Edith's friend, the mathematician Christopher C. Mihelich, has suggested that such graphs be termed hyperhamiltonian.

Edith's solution is given in the diagram for 12 nodes. It has the property that when all 12 guests are present, no spouses (the vowels) need sit next to each other. However, when any one person is omitted, all remaining spouses <u>must</u> sit next to each other.



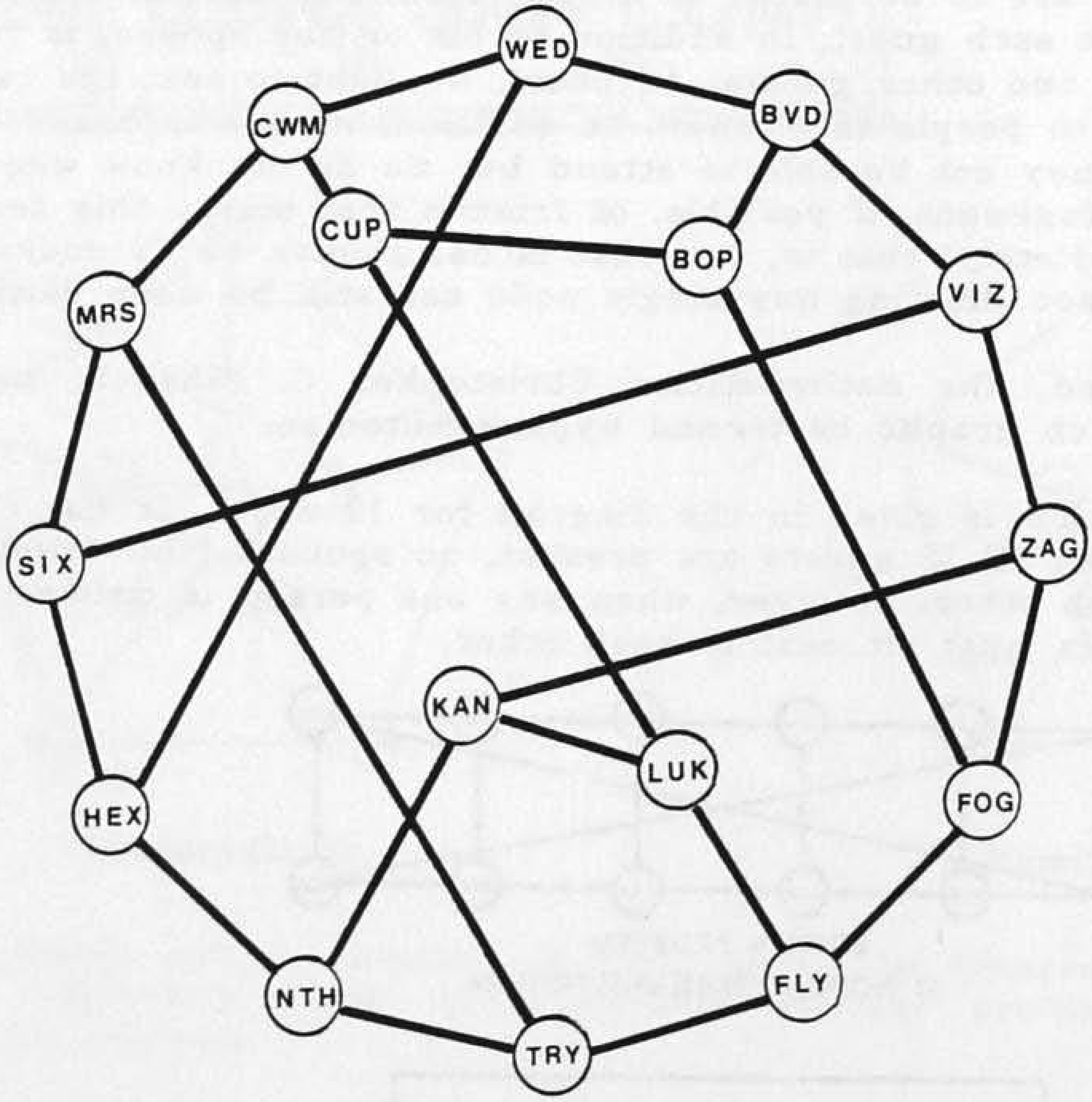
EDITH'S PROBLEM
12 NODE HYPERHAMILTONIAN



10 NODE HYPERHAMILTONIAN

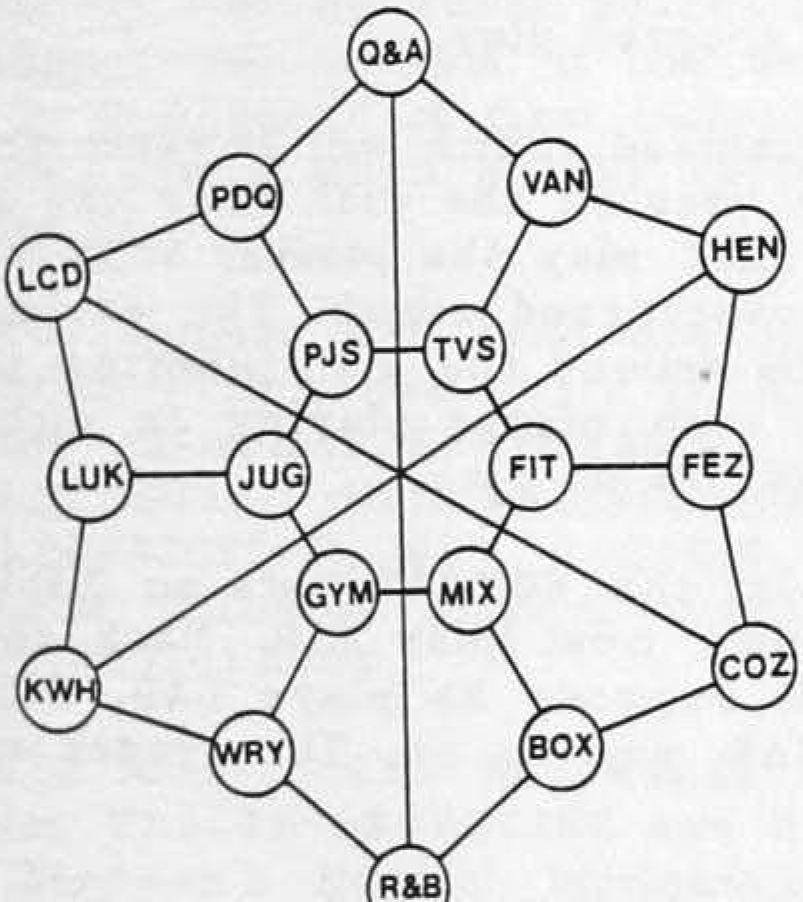
A solution to Edith's problem for five couples is illustrated in the diagram for 10 nodes. This uses the letters of SCRUBWOMAN EDITH while the 12-node uses the letters of PLY SCRUBWOMAN EDITH. The two graphs illustrate the methods needed to generate cubic hyperhamiltonian graphs for any number of nodes using a "Möbius" cross for 2x(even) number of nodes and a "cylinder" connection for 2x(odd) number of nodes. Cubic graphs are those that have three edges at every node.

The reader may like to test his "Juxtaposition Quotient" on the 16-node JQ puzzle given below. It happens to be hyperhamiltonian and is best played by placing the 16 words on scrub tiles and trying to arrange the scrubs in a circle. Once this is done, 16 more puzzles are possible by discarding any one scrub and then trying to arrange the remaining 15 in a circle. Possible answers to all these problems are in Answers and Solutions. The 16 words are all main entries in the American Heritage Dictionary (3rd edition) except for KAN, a Dutch measure in Webster's Second, and LUK, a city in Tibet.



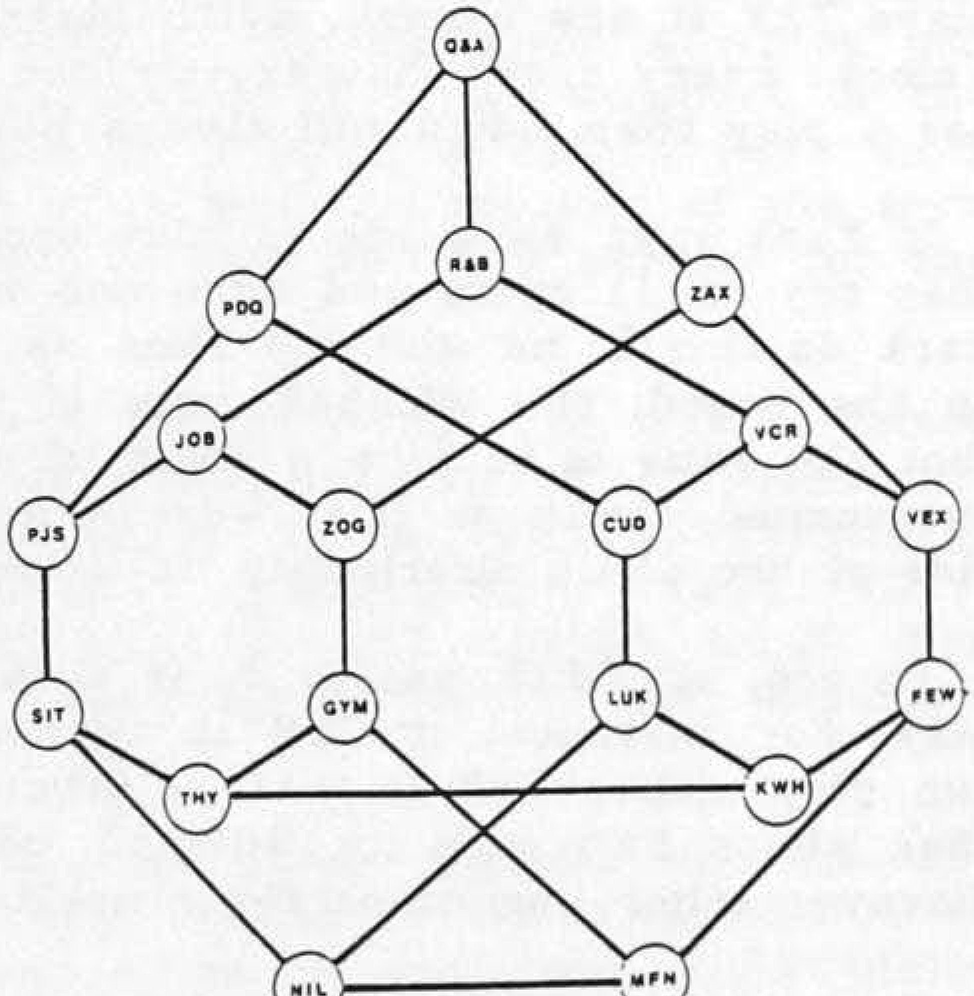
JQ PUZZLE
HYPERHAMILTONIAN

We can obtain 18-node examples if we are allowed use of the ampersand symbol along with the 26 letters of the alphabet. SNOWFLAKE is hyperhamiltonian and is a very hard puzzle played with scrub tiles. For TOPAZ, a hypohamiltonian puzzle, one could ask someone to arrange the scrubs in a circle and, of course, they will not be able to do so. When you play TOPAZ, surreptitiously palm any single scrub and in time you will be able to complete a circuit and perhaps your amazed friend will not notice that there are only 17 scrubs instead of 18.



SNOWFLAKE PUZZLE

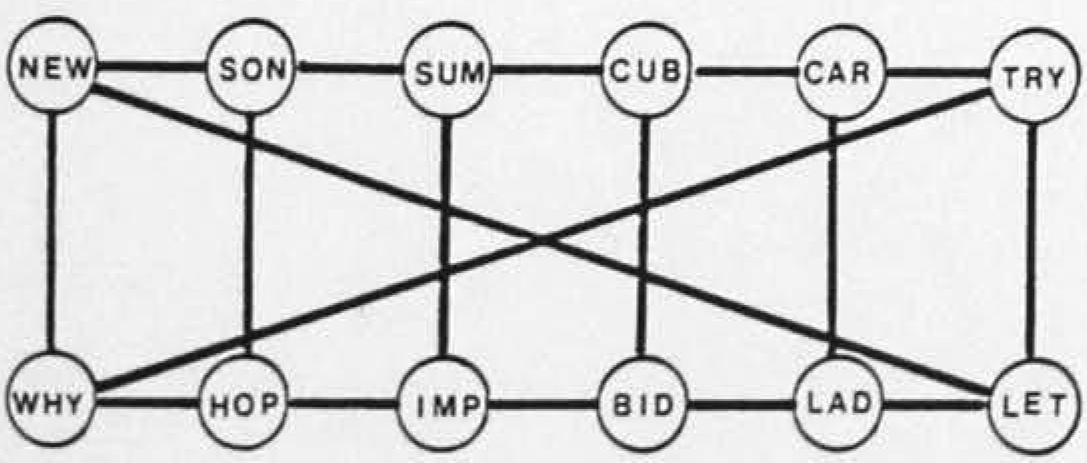
HYPERHAMILTONIAN



TOPAZ PUZZLE

HYPOHAMILTONIAN

Finally, we offer a puzzle that can be turned effectively into a very nice two-person game. Prepare 12 scrubs in accordance with the WOMBAT hyperhamiltonian graph shown. In this graph the vertical edges use the letters in WOMBAT instead of the more obvious vowels.



WOMBAT HYPERHAMILTONIAN

Edith has found an old grandfather clock dial to use as a playing board and invites her friend Mark to solve the scrub tile puzzle by placing the 12 scrubs correctly around the dial. Sooner or later, Mark finds a solution and is, no doubt, very pleased with his expertise. Edith now suggests they play a game where each will alternately place a scrub from a face-up pile until one of them has no move and loses. She generously offers Mark the choice of going first or second. If he wants to go first, Edith notes Mark's play and plays, in the diametrically opposite spot, the WOMBAT mate of Mark's play. For instance, if Mark plays TRY at one o'clock, Edith plays LET (the T from WOMBAT) at seven o'clock. Every scrub has exactly one letter from WOMBAT so that if Mark has a play then Edith will always have a counter play.

If Mark says he wants to play second instead, Edith will suggest that they try an 11 cycle and turn one scrub over on the dial. (She can let Mark do this if he wishes.) Then as her first play she places, anywhere on the board, the WOMBAT mate of the overturned scrub. She explains that the game is to form a chain of scrubs around the dial, ignoring the overturned scrub as if it were a splice, each player playing on either side of the chain alternately until someone has no play.

To win, all Edith has to do is always play the WOMBAT mate on Mark's play. For instance, if CUB is turned, Edith must play BID. Mark then can play either IMP or LAD to abut BID. Suppose he plays LAD. Edith then abuts LAD with its WOMBAT mate CAR, and so on. The reader can discover other variations for himself.

We would be remiss if we did not mention how Hamiltonian graphs got their name. The famous mathematician Sir William Rowan Hamilton (1805-1865), born of Scotch parents in Dublin, invented in about 1857 a game consisting of a regular solid dodecahedron whose 20 vertices were each labeled with the name of a well-known city. The object was that the player must find a path "all round the world" that, starting from any city, goes to every other city exactly once and ends back at the starting city. Interested readers may pursue this subject in any graph theory book or in books on the related Traveling Salesman's Problem.