LETTER SPIRALS

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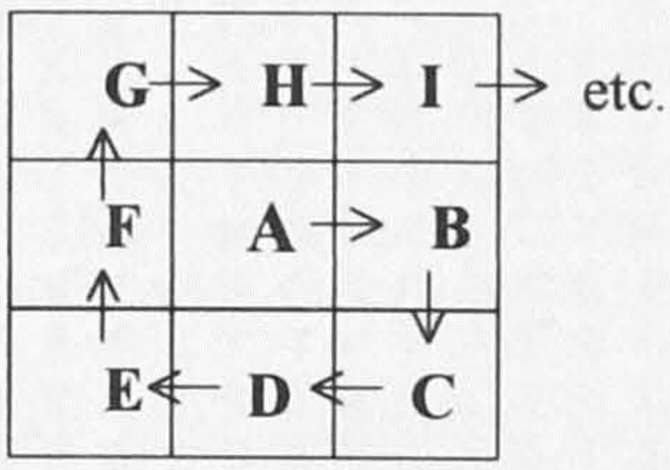
This recreation was inspired by the famous mathematical discovery of S. Ulam: if the numbers 1,2,3,4... are arranged in a spiral on a square grid, unexpected diagonal patterns appear when one looks at the prime numbers. What happens, I wondered, if we do a similar thing with the alphabet, or other sequences of letters, and look for words instead of numbers?

The first variation involves writing the alphabet in order (A to Z, then A to Z again, etc.) in an infinite spiral:



After only filling a 3x3 grid we already notice the word FAB has appeared. How many other words--and what's the longest one?--can we find, in word-search fashion, if we continue this process indefinitely?

When we have gotten as far as the E in the second alphabet

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U V W X Y Z
T G H I J A
S F A B K B
R E D C L C
Q P O N M D
          E
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the first four-letter word, ACME, appears diagonally (starting on the very first square of the spiral!). By this point we also have the three-letter words ADO, HAD, and RED, and a number of two-letter words.

The first five-letter word, SIGMA, appears (diagonally) when we have progressed to a 19x19 square centered on the initial A. Suddenly, new words stop appearing! To see why this happens, a smidgen of spiral mathematics is needed. Consider the spiral points as being numbered starting with zero. Denote by p(n) the number assigned to the nth point on the diagonal leading southwest from the initial A (with n=0 corres-

ponding to the A square). Then, using the fact that the first n consecutive odd numbers add up to n^2, it is easy to derive $p(n) = (2n)^2$ and, further, that the four points on the corner of each "shell" of the spiral (of which this is one) are numbered $(2n)^2 + k(2n+1)$, for k=0,1,2,3. Now note that the letter in a square numbered m is simply m modulo 26, with 0=A, 1=B, ... 25=Z. But wait! This sequence p(n) of corner point numbers, when reduced modulo 26, repeats with a period of 13. This means that every 13 shells, the spiral repeats. So, one we get sufficiently far into the spiral to avoid edge effects near the center (which doesn't take very long), we will find no new configurations of letters and hence no new words.

Here is a 40x40 portion of the spiral, with the initial A circled in the center. The two sets of lines (located on southwest corner points of two shells) show the first place where repetition starts to occur.

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A B C D E F G H I J K L M N O P Q R S T U V W X Y Z A B C D E F G H I J K L M N
Z G H I J K L M N O P Q R S T U V W X Y Z A B C D E F G H I J K L M N O P Q R O
Y F U V W X Y Z A B C D E F G H I J K L M N O P Q R S T U V W X Y Z A B C D S P
X E T Q R S T U V W X Y Z A B C D E F G H I J K L M N O P Q R S T U V W X E T Q
W D S P U V W X Y Z A B C D E F G H I J K L M N O P Q R S T U V W X Y Z Y F U R
V C R O T G H I J K L M N O P Q R S T U V W X Y Z A B C D E F G H I J A Z G V S
U B Q N S F A B C D E F G H I J K L M N O P Q R S T U V W X Y Z A B K B A H W T
T A P M R E Z C D E F G H I J K L M N O P Q R S T U V W X Y Z A B C L C B I X U
S Z O L Q D Y B M N O P Q R S T U V W X Y Z A B C D E F G H I J C D M D C J Y V
R Y N K P C X A L E F G H I J K L M N O P Q R S T U V W X Y Z K D E N E D K Z W
Q X M J O B W Z K D E F G H I J K L M N O P Q R S T U V W X A L E F O F E L A X
P W L I N A V Y J C D M N O P Q R S T U V W X Y Z A B C D Y B M F G P G F M B Y
O V K H M Z U X I B C L C D E F G H I J K L M N O P Q R E Z C N G H Q H G N C Z
N U J G L Y T W H A B K B A B C D E F G H I J K L M N S F A D O H I R I H O D A
M T I F K X S V G Z A J A Z G H I J K L M N O P Q R O T G B E P I J S J I P E B
L S H E J W R U F Y Z I Z Y F U V W X Y Z A B C D S P U H C F Q J K T K J Q F C
K R G D I V Q T E X Y H Y X E T Q R S T U V W X E T Q V I D G R K L U L K R G D
J Q F C H U P S D W X G X W D S P U V W X Y Z Y F U R W J E H S L M V M L S H E
I P E B G T O R C V W F W V C R O T G H I J A Z G V S X K F I T M N W N M T I F
H O D A F S N Q B U V E V U B Q N S F (A) B K B A H W T Y L G J U N O X O N U J G
G N C Z E R M P A T U D U T A P|M R E D C L C B I X U Z M H K V O P Y P O V K H
F M B Y D Q L O Z S T C T S Z O|L Q P O N M D C J Y V A N I L W P Q Z Q P W L I
E L A X C P K N Y R S B S R Y N|K J I H G F E D K Z W B O J M X Q R A R Q X M J
D K Z W B O J M X Q R A R Q X M|L K J I H G F E L A X C P K N Y R S B S R Y N K
C J Y V A N I L W P Q Z Q P W V U T S R Q P O N M B Y D Q L O Z S T C T S Z O L
B I X U Z M H K V O P Y P O N M L K J I H G F E D C Z E R M P A T U D U T A P M
A H W T Y L G J U N O X O N M L K J I H G F E D C B A F S N Q B U V E V U B Q N
Z G V S X K F I T M N W V U T S R Q P O N M L K J I H G T O R C V W F W V C R O
Y F U R W J E H S L M L K J I H G F E D C B A Z Y X W V U P S D W X G X W D S P
X E T Q V I D G R K J I H G F E D C B A Z Y X W V U T S R Q T E X Y H Y X E T Q
W D S P U H C F Q P O N M L K J I H G F E D C B A Z Y X W V U F Y Z I Z Y F U R
V C R O T G B E D C B A Z Y X W V U T S R Q P O N M L K J I H G Z A J A Z G V S
U B Q N S F A Z Y X W V U T S R Q P O N M L K J I H G F E D C B A B K B A H W T
T A P|M R E D C B A Z Y X W V U T S R Q P O N M L K J I H G F E D C L C B I X U
S Z O|L Q P O N M L K J I H G F E D C B A Z Y X W V U T S R Q P O N M D C J Y V
R Y N|K J I H G F E D C B A Z Y X W V U T S R Q P O N M L K J I H G F E D K Z W
Q X M|L K J I H G F E D C B A Z Y X W V U T S R Q P O N M L K J I H G F E L A X
P W V U T S R Q P O N M L K J I H G F E D C B A Z Y X W V U T S R Q P O N M B Y
O N M L K J I H G F E D C B A Z Y X W V U T S R Q P O N M L K J I H G F E D C Z
N M L K J I H G F E D C B A Z Y X W V U T S R Q P O N M L K J I H G F E D C B A
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The word SIGMA is shown in bold face not too far from the center square. For completeness, here are all 98 of the reasonably common words we have been able to find:

ad ah ai am an as at aw ax ay be bo by do eh em fa go ha he hi ho id if in is it la me my no nu od of oh on or ow ox pi re sh si so ti to up us we xi

ado ale ass aye bah den fad fed fit fur had hod hut ice job keg lax maw may nil orc pat red rot rut she sue tap tor van woo yak yam yes you

acme ayes dene derm goes hazy huts navy ruts

sigma

Since a sigma in mathematics is used to represent a sum, this seems like a good time to sum up this discussion and move on to the next, more challenging variant. In this one we mimic Ulam's mathematical spiral even more closely, but instead of putting 1,2,3... in a spiral, we put the words ONE, TWO, THREE, etc., following the same path as shown above. What words can we find in the resulting word-search grid, and--the most intriguing question--what's the longest one?

First, we need to note that the answers may be different (at least when we get to far-flung areas of the spiral) depending on whether we use the American or British numbering names (billion vs. milliard, etc.), We have only examined the American spiral; logophiles across the pond are encouraged to explore the alternate version.

Since the letter frequencies in the spiral are much closer to normal English we are not surprised to find many more words than in the previous case. Here is the initial 9x9 portion of the spiral:

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E N S I X T E E N
E V E N T W E L V
T E I X S E V E E
F L S T H R E N T
I E E O O N E E H
F N V W T E F I I
N E I F R U O G R
E T E N I N T H T
E T R U O F N E E
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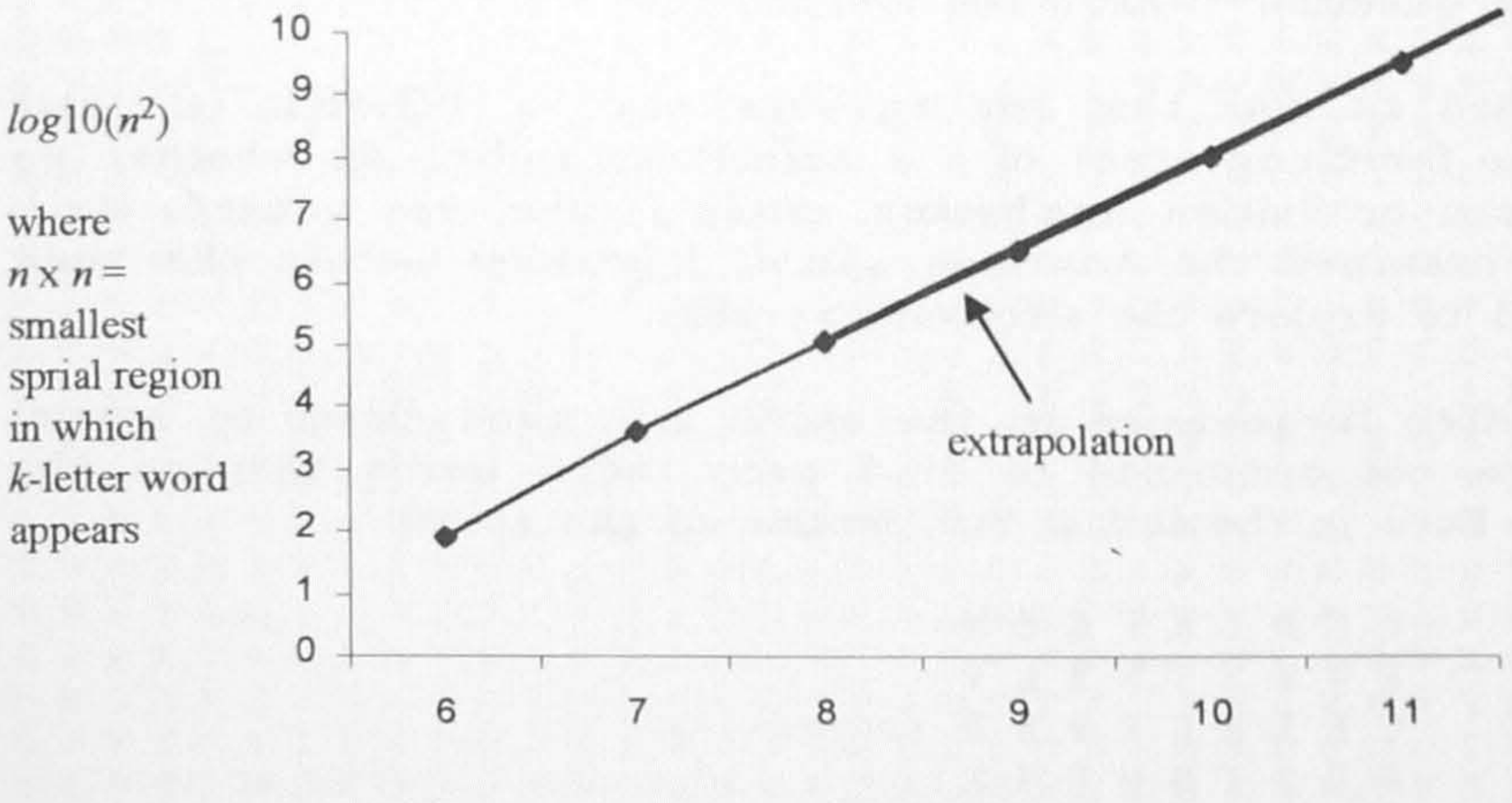
We see a 6-letter word (REIVES, ascending from the bottom row), several 5-letter words such as EVENT (on the second row), and a number of smaller ones. The first 7-letter word, ORDERED, does not appear until we get to 63x63. (Number words or obvious derived forms such as EIGHTS or TWELFTH are considered uninteresting and ignored.) As we continue to look further and further, new words are harder to come by but they

do keep appearing. At size 343x343, we find the first eight-letter word, DEHORNED.

We searched up to 500x500 for words of all sizes, and found a total of 2733 words with two or more letters: 41 2-letter, 332 3-letter, 944 4-letter, 1019 5-letter, 360 6-letter, 36 7-letter, and one 8-letter word. Here are the 7-letter words:

denudes desired dessert detente dryness earthen endurer entwine
evenest finesse heftier hottest hounded ideated insight nations
newtons ordered outdoor outrode reddens redound riotous seethes
sennits taunter teethes tendons tenture tenures terrene tertian
testier tressed tsetses unwound

If we continue to search outwards, are there other eight-letter words, or even longer ones? It seems likely that the answer is yes, so we would like to estimate how far one might need to search. To do so, let n(k) be the area of the smallest portion of the spiral that contains a k-letter word. A heuristic argument suggests that k is proportional to the logarithm of n(k). Plotting the first three good data points we have (for k=6,7,8), as shown in the graph below, we discover that this assumption is a good one since the points nearly lie on a straight line.



Extrapolating this line outward, we find that the first 9-letter word should occur in a region of size about 1800x1800 (since 1800 is approximately the square root of $10^{6.5}$, where 6.5 is the value on the y-axis corresponding to x=9). A 10-letter word might occur around 10000x10000, and an 11-letter word when we have gotten to 55000 on a side.

Encouraged by this analysis, we set our computer searching overnight in a 2500x2500 spiral. We found no fewer than five 9-letter words: INTERNEES, UNENTERED, INSURGENT, DETERRENT and INTENDANT. An additional computer run determined that there are no 10-letter words at size 4000x4000.

What 10-letter words are the most likely to occur? For all number words from 1 to 999,999, the letter-frequency statistics are:

E 0.141	N 0.119	D 0.102	T 0.097
H 0.089	U 0.071	R 0.067	I 0.058
O 0.053	S 0.049	Y 0.036	F 0.032
V 0.023	A 0.022	G 0.013	W 0.013
X 0.013	L 0.001		

Based on these statistics, longer words that we might hope to find include UNDETERRED, UNTETHERED, INTENTIONED and RHODODENDRON.

The gauntlet is hereby thrown to Word Ways readers: can you find 10-letter or longer words in the ONE, TWO, THREE spiral?

MARTIN'S MENTAL "MAGIC"

Remember "Mysterious Precognitions", a collection of word tricks by Martin Gardner in the August 1998 Word Ways? These can be found in his new paperback *Mental Magic*, containing 89 easy-to-perform tricks involving words, arithmetic, cards, coins and dice. Many tricks involve apparent precognition on the part of the performer, but in most cases there are mathematical (or lexical) explanations why the outcome is certain to occur. (Some tricks are merely surprising, such as the fact that the fourth root of 2143/22 is very nearly equal to pi.)

Although most of the explanations are well-known, some call for further lexical research. Most noteworthy in this regard is the convergence that occurs when one picks an n-letter word in running text, counts forward n words to a new one, and repeats this process several (say, ten) times. No matter where you start, eventually you are likely to end up on the same word. And in his "Fold and Trim" trick (also mentioned in Aug 1998 Word Ways), what eight-letter words lead to anagrammable words on both the four face-down cards and the four face-up ones? And what about other foldings besides the 4x2 one?