ORCHARD PROBLEMS IN WONDERLAND

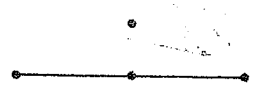
JEREMIAH FARRELL
Indianapolis, Indiana

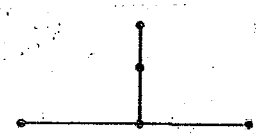
> Your aid I want, nine trees to plant
> In rows just half a score;
> And let there be in each row three.
> Solve this: I ask no more.
>
> -John Jackson

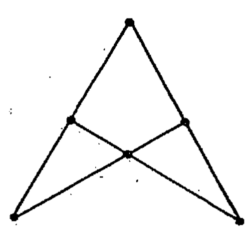
The poem refers to a "tree plant" problem now usually referred to as an Orchard problem. The first reference to Orchard was in an 1821 book called *Rational Amusement for Winter Evenings* by John Jackson. He asked for 9 trees to be planted in 10 rows of three. Ten trees was later proven to be the maximum number of rows for n=9. According to Martin Gardner in his *Time Travel and Other Mathematical Bewilderments,* Freeman 1988, the problem of finding the maximum number of rows for a given number n of trees is still unsolved for arbitrary n. He adds that when the rows are to have K=3 trees each "the problem not only becomes interesting but also is related to such mathematical topics as balanced-block designs, Kirkman-Steiner triples, finite geometries, Weierstrass elliptic functions, cubic curves, projective plans, error-correcting codes, and many other aspects of significant mathematics."

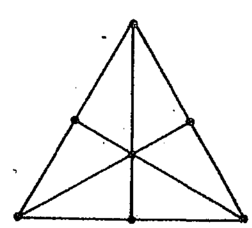
From Gardner's book we list some of the known optimal designs.

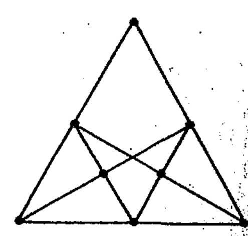

$n = 3, r = 1$

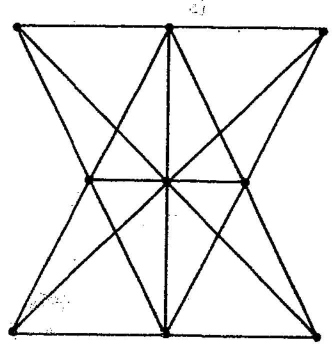

$n = 4, r = 1$

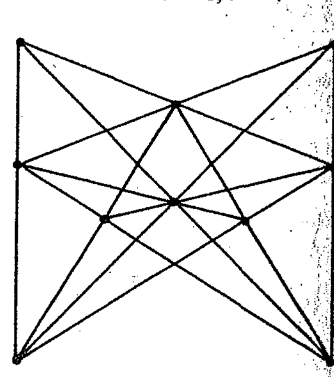

$n = 5, r = 2$


$n = 6, r = 4$


$n = 7, r = 6$

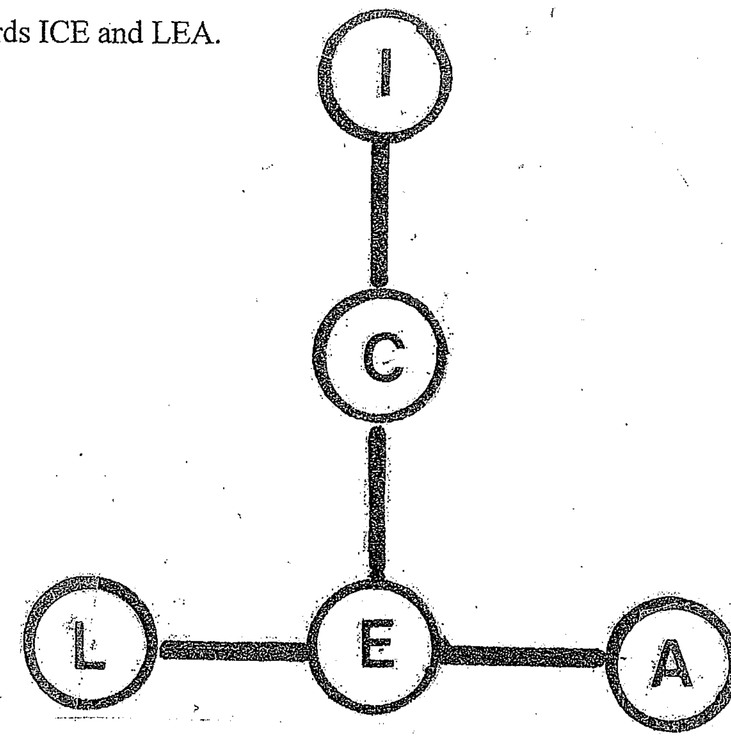

$n = 8, r = 7$


$n = 9, r = 10$

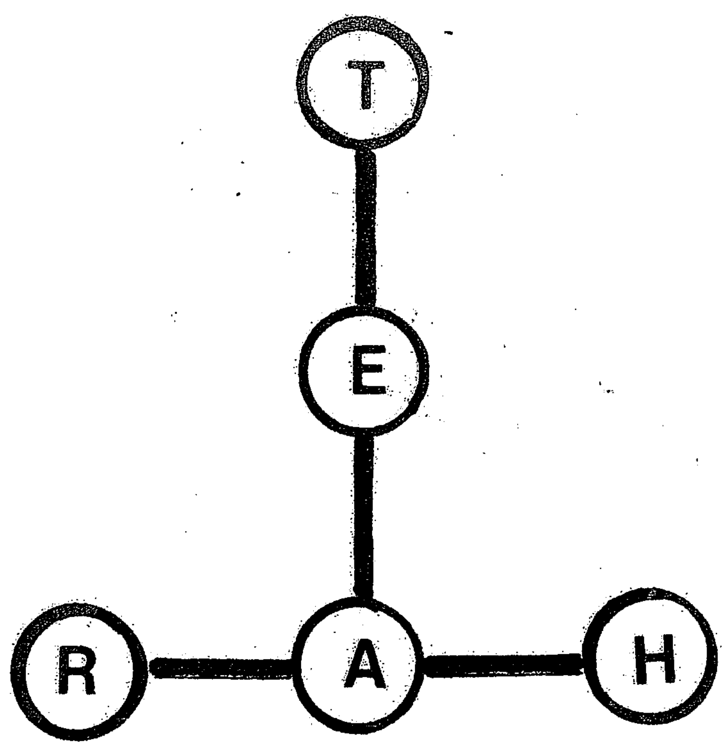

$n = 10, r = 12$

Instead of trees in rows of three we will use letters in the name of characters from the works of Lewis Carroll's "Alice". (n,r) will represent the number of nodes (letters) and the maximum number r of rows of 3. Several examples follow.

ALICE a (5,2) yielding the two words ICE and LEA.

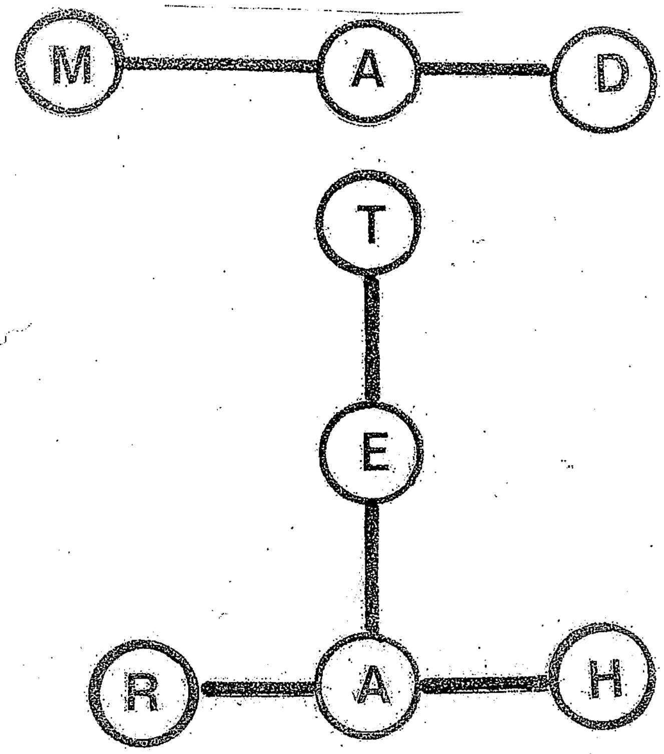


Also using the different letters of HATTER, we have

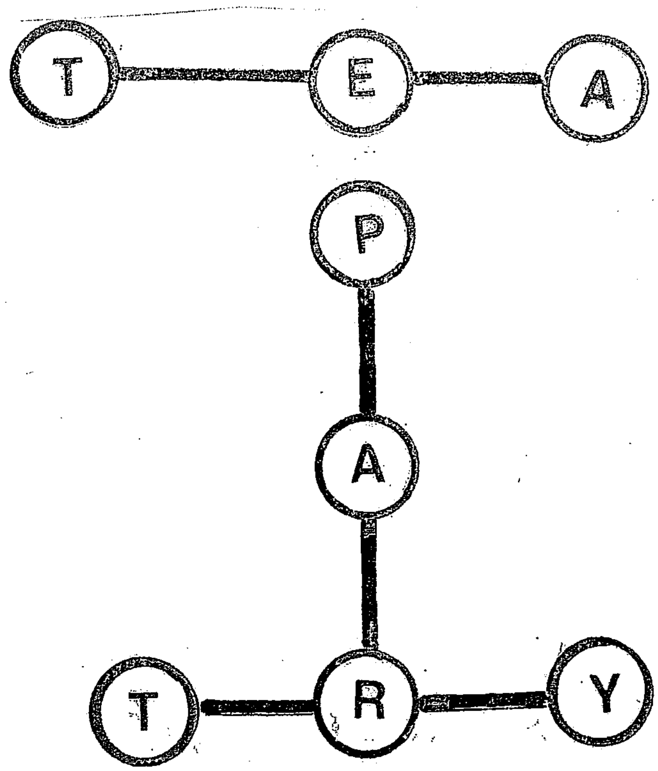


Yielding TEA and RAH.

We can also obtain MAD HATTER using (3,1) and (5,2) graphs.

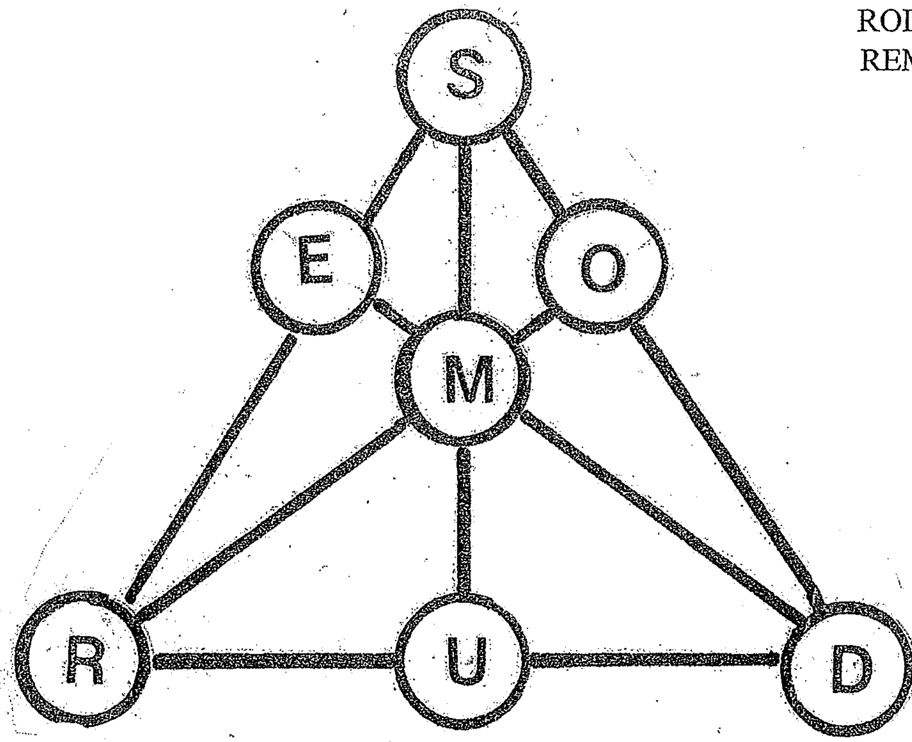


Also at the TEA PARTY



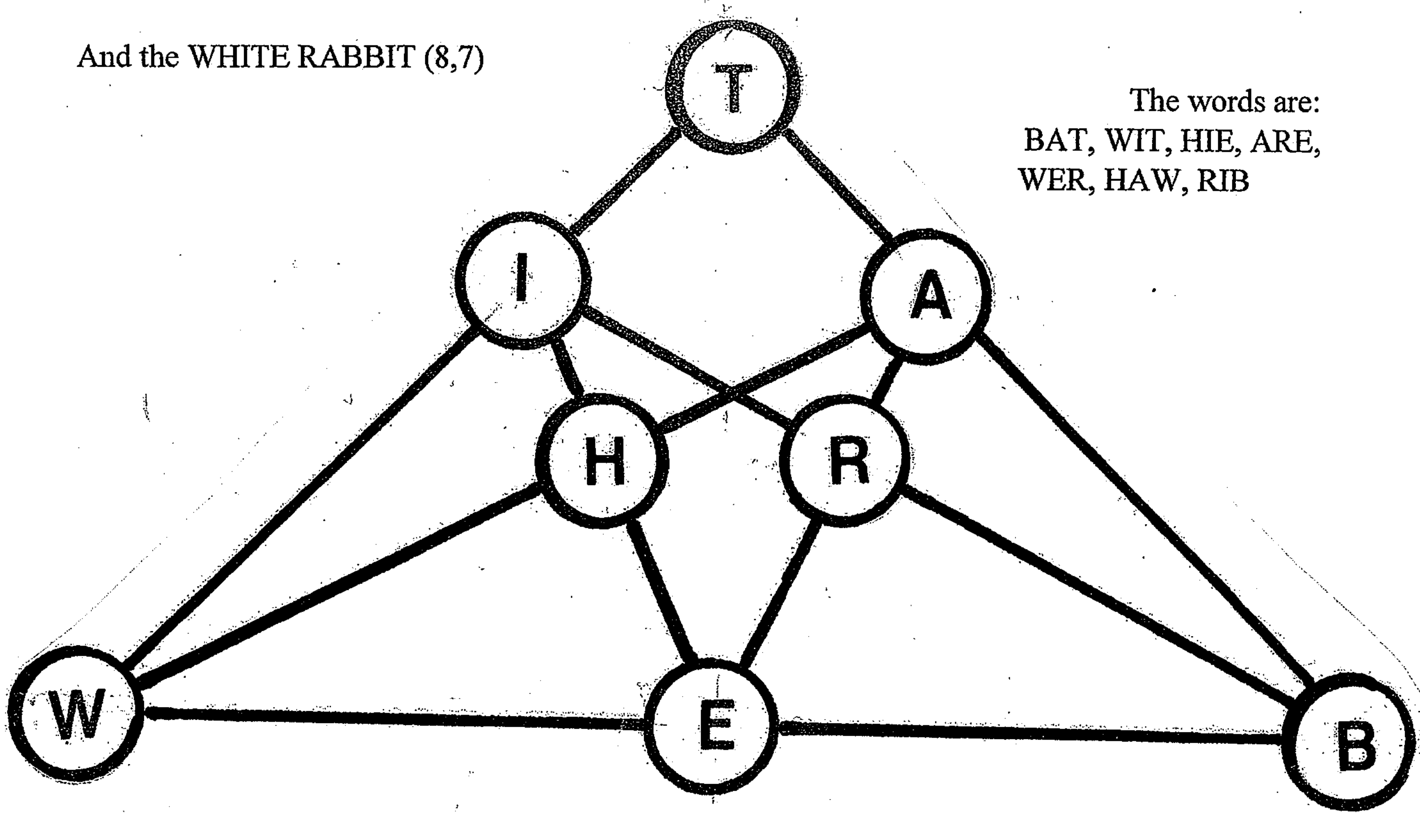
Also there was the DORMOUSE (7,6)

The words are:
ROD, OMS, SUR,
REM, EDS, MUD



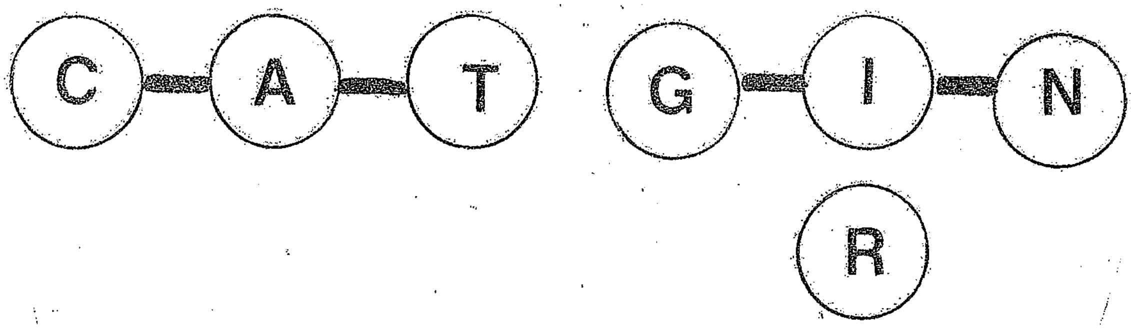
And the WHITE RABBIT (8,7)

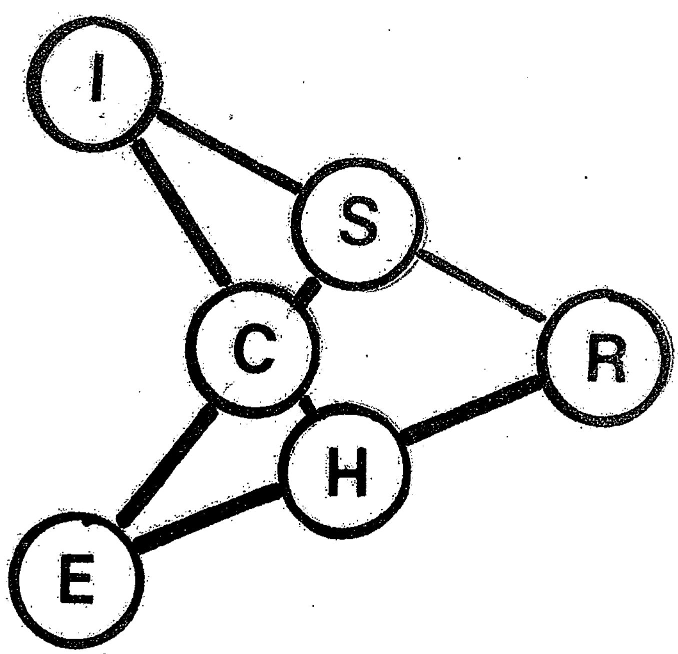
The words are:
BAT, WIT, HIE, ARE,
WER, HAW, RIB



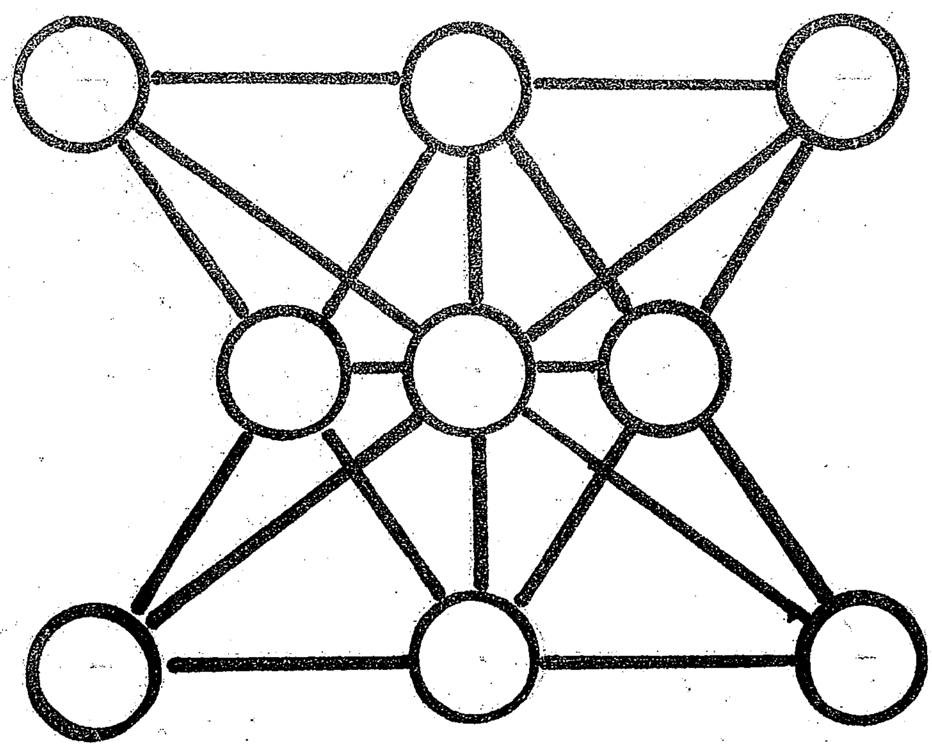
We don't want to forget the CHESHIRE (6,4) CAT (3,1) with his GRIN (4,1).

The words are:
SIR, HER, SEC, CHI



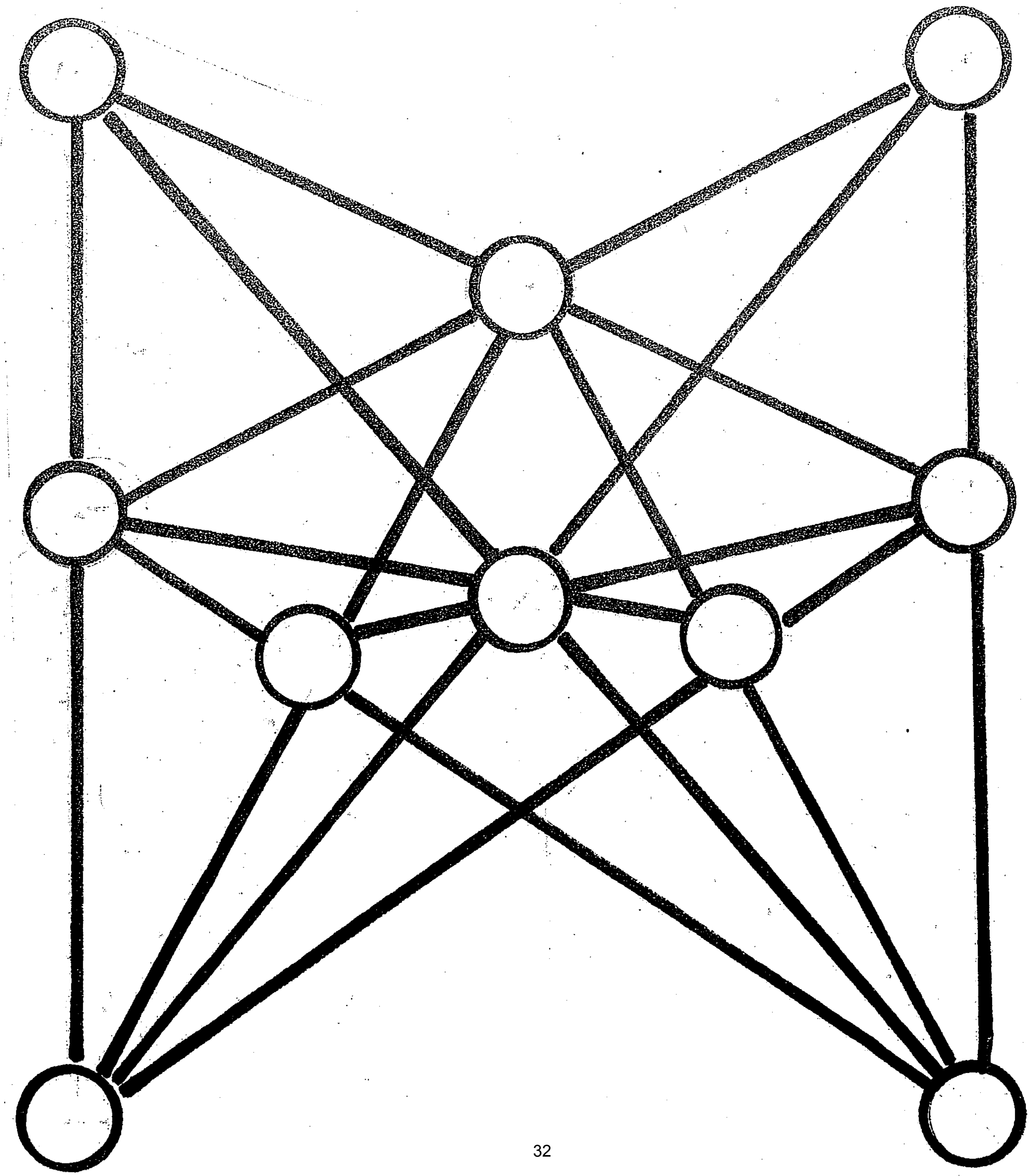


And what about LEWIS CARROLL? We use Jackson's (9,10) here obtaining the 10 words ALE, COW, ICE, ORE, RAW, SAC, SEW, SIR, SOL (Hurok impresario), WIL (Wheaton) actor. Can the reader fill in this graph to form the 10 words? Or do better?



For a (10,12) we offer these 12 words which use the 10 different letters of ALICE IN WONDERLAND.

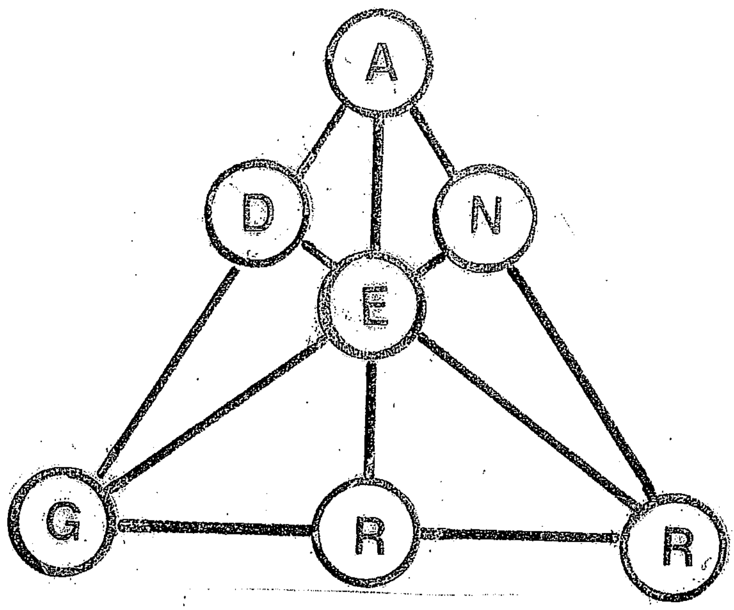
CAN, CEL, COW, DIN, LAR (household god), NEW, NOR, OLD, RED, RIC (Ocasek, singer), WAD, WIL. Can the reader fill in the grid?



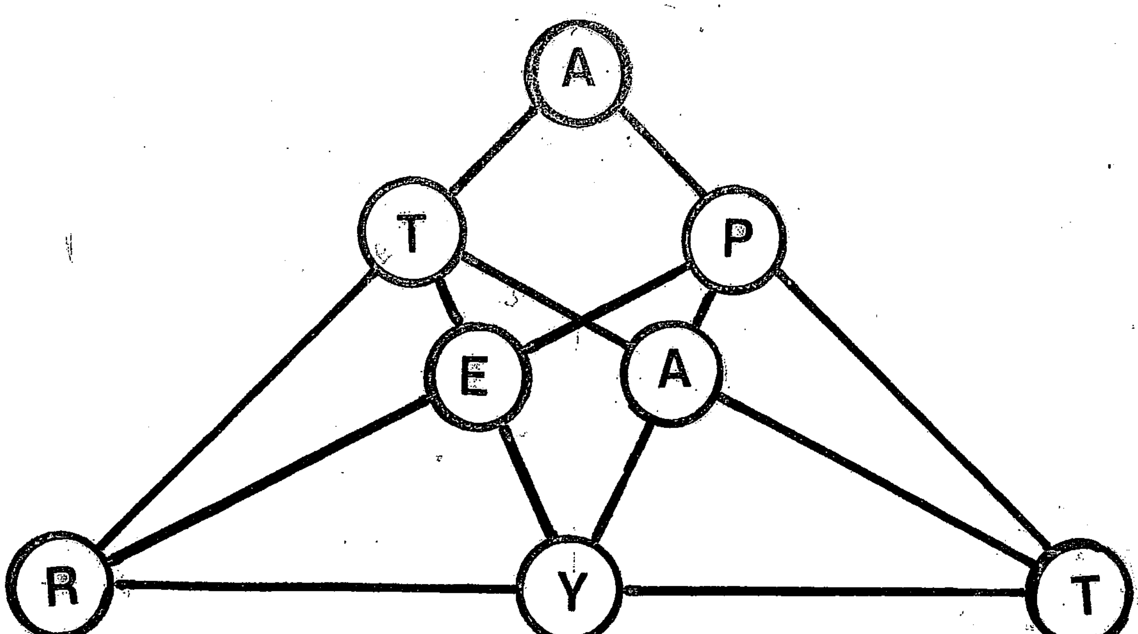
Our answers to the (9,10) and (10,12) appear later. On both of these grids a tic-tac-toe like game can be played. Interestingly enough they are both first person wins. Perhaps the reader can find them.

No doubt the reader is quite familiar with scrambled or jumbled letter puzzles where one is to find the proper arrangement into a legitimate word from the scrambles. In constructing our puzzles the real words are to be anagrammed so as to fit in the geometry instead.

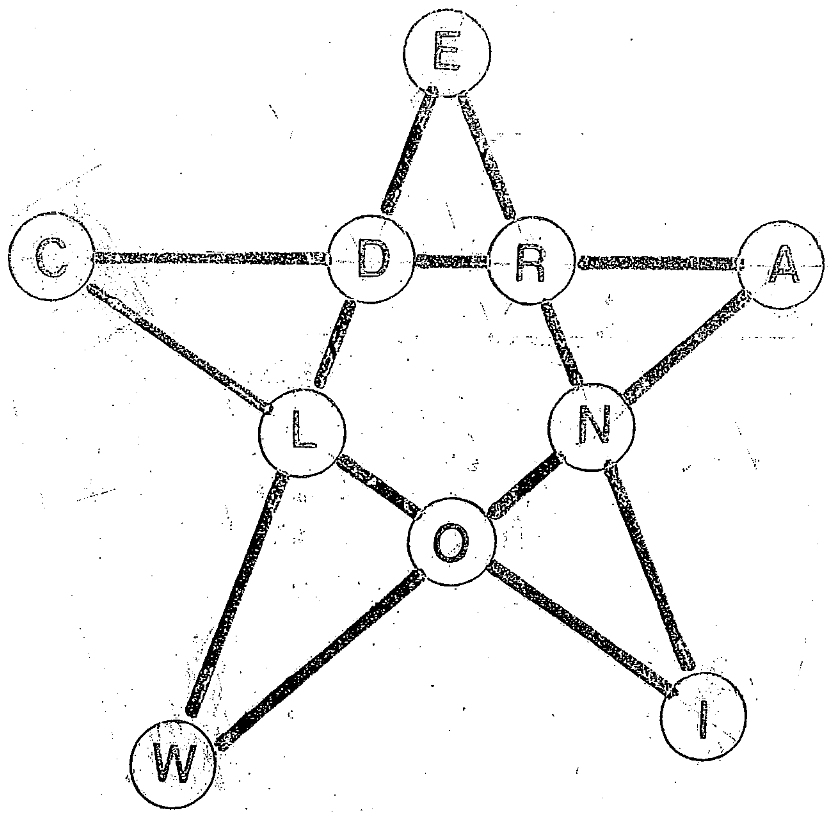
We usually use only the different letters of a name in our puzzles and all the letters can be used. Two example are GARDNER (7,6) and TEA PARTY (8,7). For Gardner the 6 words are GAD, ARE, RAN, RED, ENG (represents the sound of English NG), and GRR (dog's bark).



For TEA PARTY the words are ART, TAP, YET, TAT, PER, PAY and TRY.



Our final example is an Orchard extension. Instead of rows of three we could insist on rows of 4 (or more). This problem is also unsolved in general but the 10 letters of ALICE IN WONDERLAND can be arranged four in a row on a maximum of five lines as follows.

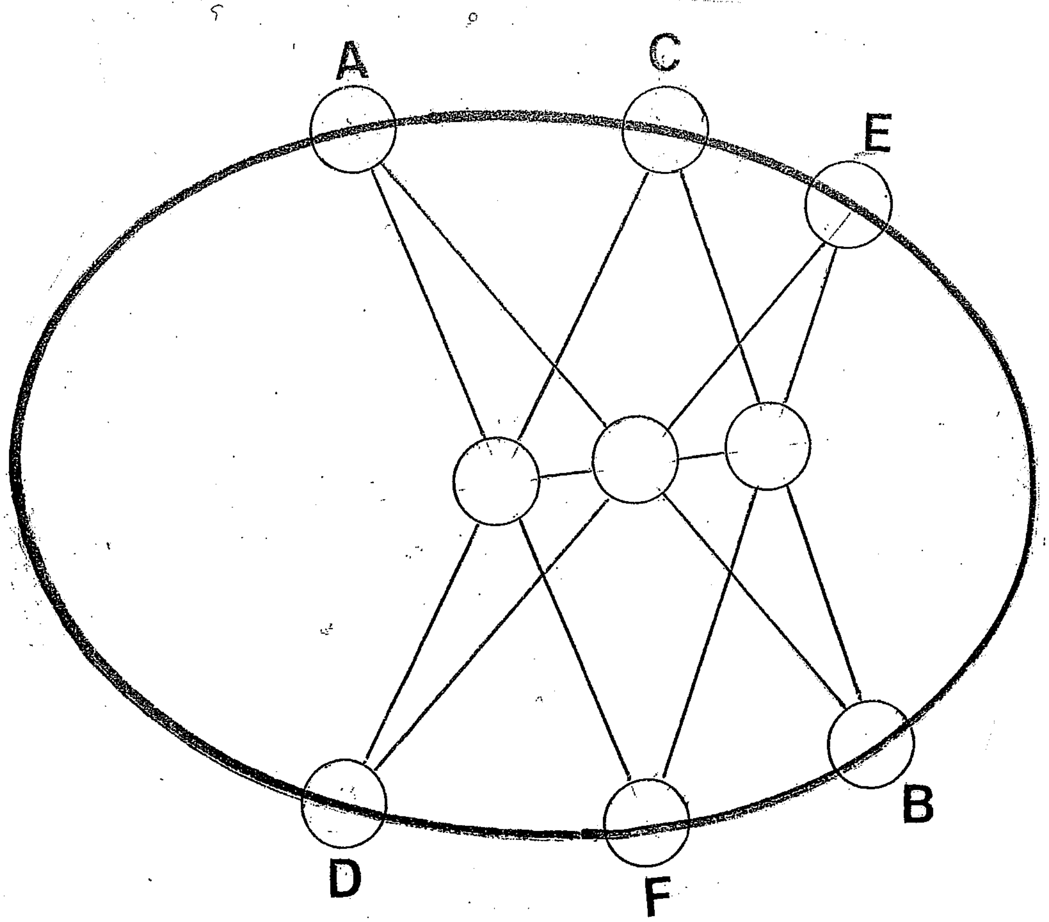


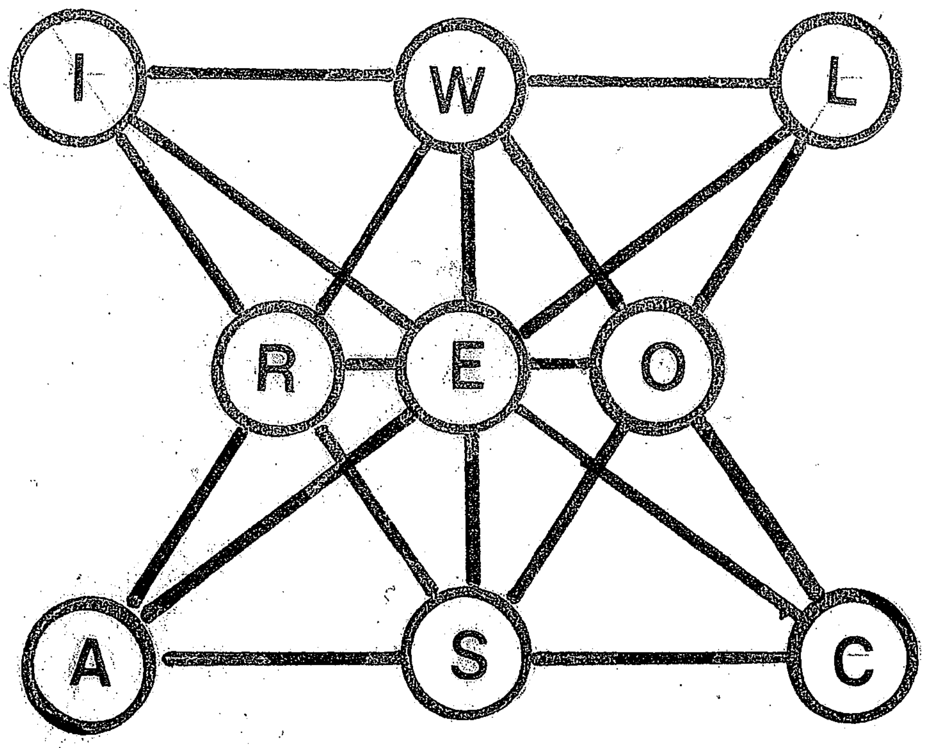
Notice that the five rows anagram into REIN, COIL, WELD, CARD, ANOW (Milton's enough). Moreover there are ten triangles: five small: CLD (Charles Lutwidge Dodgson), ION, LOW, RAN, RED and five large: WAD, OCA (S. Am. Wood Sorrel), NEW, LIE, RIC (Ocasek).

A nice game can be played on this last diagram. Two players alternately choose a node and the first to obtain any one of the ten triangles wins.

The last grid is also one example of a 10-3 configuration graph. These are described in *Tribute to a Mathemagician* published by AK Peters in 2005 in the article "Configuration Games" by Jeremiah Farrell, Martin Gardner and Thomas Rodgers. That article explains how first wins in this game and also the (9,10).

It is of some interest to note that our (9,10) diagram extends from a 9-3 configuration due to Pappus (300 A.D.) where three points can be arbitrarily placed on two lines and a generalized hexagon traced between the six. Pappus showed that the interior three intersections are always collinear. Some 13 centuries later, at the age of 16, B. Pascal showed this was true not only for lines but for any conic section too. The following illustration demonstrates this for an ellipse and the hexagon ABCDEF. Note in this case that the tenth added line CF does not coincide with the interior center point as it does in our symmetric (9,10).




I
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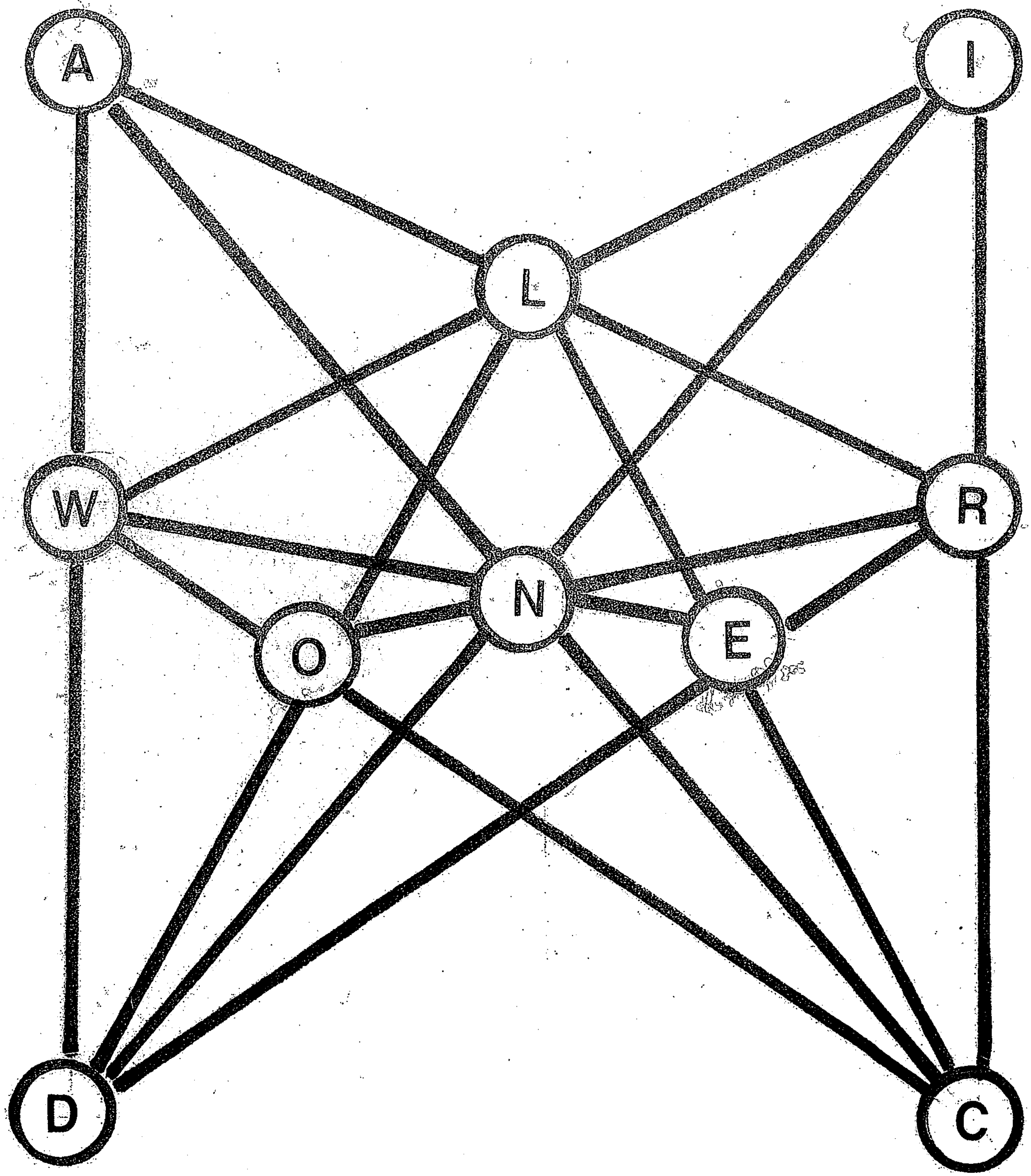
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