WHAT'S REALLY MAGIC
ABOUT MAGIC SQUARES

by
Jeremiah Farrell

We propose several new concepts about magic squares suitable for all sorts of audiences – even for television. Our first example is an adaptation of an old trick, which is an ideal presentation for classrooms of all ages.

Presentation: "I am going to fill this 4x4 grid with integers of your choice. Roll a die for our starting number and tell me what it is. If you don't have any dice you may do this mentally."

Suppose they roll a 5. You write 5 on a tablet and secretly write 50 on the tablet's reverse as it is placed on the chalk tray of the blackboard. The class is then asked to choose an order of the grid's rows and you write the numbers in order in their choice of rows. Perhaps this is the result.

9	10	11	12
17	18	19	20
5	6	7	8
13	14	15	16

"Now someone choose any number on the grid." Suppose they select 10. "I circle 10 and cross out the rest of the numbers in the 10's row and column. Now someone else choose a number not already crossed off." This is repeated until four numbers are selected. Suppose the four are 10, 16, 5 and 19.

"Now someone please add these four." No matter what four they choose the sum will always be 50 for a 5 start and your unexpected reveal of 50 on the back of the tablet will usually astonish the audience.

This is our version of an effect described by Martin Gardner in Chapter 2 of his book *Mathematical Puzzles and Diversions* (1962 Simon & Schuster), a collection of his "Mathematical Games" columns from *Scientific American*. It also appears in one form as the "Egyptian Fortune Telling Tablet" by Subir Kr. Dhar in issue 24, Dec. 1973 of Sam Dalal's *Swami.*

Any n x n grid with the integers consecutively placed in the rows (or in the columns) will always have the same sum in our selection process. For any start integer m, positive, zero or negative, the final sum for an n x n grid will be $m \cdot n + n(n^2-1)/2$.

The most popular n's are n=3, 4, or 5 and $m \cdot n$ will thus be added respectively to 12, 30, or 60. In the above example m = 5 so we add 5(4) = 20 to 30 to obtain the final 50. Negatives are OK too. If a 2 is rolled and the roller chooses to make it -2, we subtract 8 from 30 to obtain a new constant of 22 and count in order -2, -1, 0, 1, and so on.

It is of some value to give the class the problem of explaining fully why this technique always works. Perhaps they can notice that the rows and columns each differ by the same numbers so that our choice will always choose the same 2n numbers.

Other methods can expand our magic to more common squares. For example in the 4x4 case we start by using the numbers in this grid whose construction is straight forward.

4	5	6	7
3	4	5	6
2	3	4	5
1	2	3	4

These numbers sum to 4^3 and can be rearranged into a remarkable magic square.

5	5	4	2	5	5
5	1	6	4	5	1
4	6	3	3	4	6
2	4	3	7	2	4
5	5	4	2	5	5
5	1	6	4	5	1

The bold 4x4 outlined grid represents one solution. The border cells mark the start of a tiling of the plane by the solution grid. Alternately it can be regarded as a folding of the 4x4 into a torus. There are 36 sets of four squares that sum to the magic constant 16.

Before we comment further about this square we perform some magic using it. First we hide the number 16 under a slip of paper. Then ask a member of our audience to choose any number. Suppose he selects and circles a 5. Other members are asked to form a square with the circled 5 as a corner and then add the four numbers. The four always total 16!

The square is also magic with constant 16, not only on the rows and columns but also on all eight diagonals including the broken ones. Note also the four corners of any 2x4 rectangle for instance.

It is a fact that for all n equaling 3 or greater our method will always produce an n x n magic square with constant n^2 at least on rows, columns, and the two main diagonals.

For n=3 this grid is used.

3	4	5
2	3	4
1	2	3

We leave to the reader the problem of using these nine numbers in a 3 x 3 magic square with constant $3^2 = 9$.

For n = 5 we like the "Greek Cross" square constructed from the following grid.

5	6	7	8	9
4	5	6	7	8
3	4	5	6	7
2	3	4	5	6
1	2	3	4	5

9	3	6	4	3	9	3
7	1	7	6	4	7	1
4	7	5	4	5	4	7
2	8	2	5	8	2	8
3	6	5	6	5	3	6
9	3	6	4	3	9	3
7	1	7	6	4	7	1

The bold-face 5x5 grid is a solution. The outer border describes the folding of the grid into a torus so as to identify 25 small Greek crosses that each sum to the magic constant 25. The 25 are each centered at one of the 25 entries in the 5x5 grid.

A Greek Cross has orientation of one of two types: 

For example the two crosses centered at 1 are:

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        3                              9     6
  7     1   7          OR                 1
        7                              4     5
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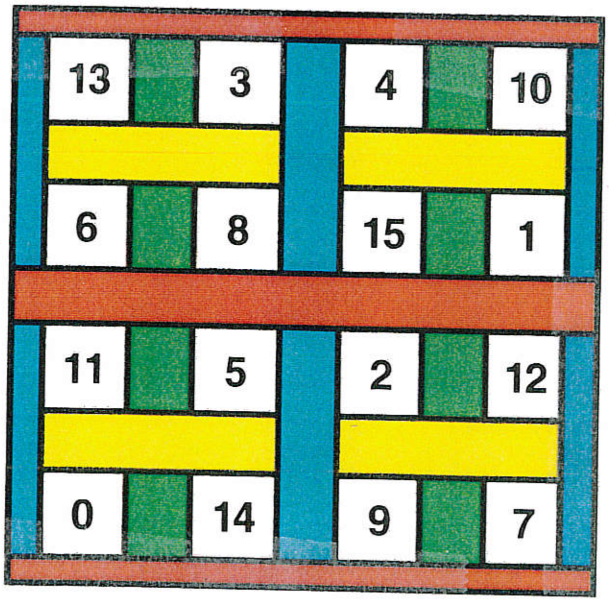
Before pointing out for the audience that this square is magic on all rows, columns and diagonals we ask volunteers to select a Greek Cross and add the five numbers. The sum will always be our previously predicted 25, the magic constant of the square.



The 4x4 square above is our modification of one using 16 consecutive numbers.

Our solution requirements make this puzzle one of what is called by Dame Kathleen Ollernshaw a most-perfect magic square. This remarkable lady published when she was 87 years old (*Most-Perfect Pandiagonal Magic Squares*, The Institute of Mathematics and its Applications, 1998, Great Britain, University Press, Cambridge) a complete solution set for all 4k=n such squares. With her co-author David Brée, this was the first time a complete enumeration of an infinite subset of magic squares was completed.

Some mathematical background. There are (16!) / (4!)(3!)(3!)(2!)(2!) = 6054048000 ways of filling the grid with the 16 tokens that will look different to the eye. If we don't count reflections and rotations as different this reduces to 756756000 different placements.

The generic term "tokens" usually refers to the numbers that are used in our squares but need not always to be so. We later will employ playing cards and even colors and shapes as tokens.

In his November 1999 *Scientific American* column "Mathematical Recreations" Ian Stewart notes that Ollerenshaw's "most perfect" requirement is for any magic square of size 4n x 4n with the property that any two-by-two block of adjacent entries have the same sum. He adds "The discovery of the (complex) formula and its proof, leads deeper into combinatorics, so I'll stop here, except to say that for the doubly even orders of 4, 8, 12 and 16, the numbers of different most-perfect magic squares are 48; 368, 640; 2.22953×10^{10} and 9.322433×10^{14}."

Martin Gardner in vol. 395, 17 Sept. 1998 of *Nature* also reports on the achievement. " Dame Kathleen Ollerenshaw, one of England's national treasures, has solved a long standing, extremely difficult problem. . . ".

For the 4 x 4 Gardner points out that every broken diagonal sums to 30 as well as every row and column. Also, every 2 x 2 square sums to 30 and also any two cells a diagonal hop apart add up to 15.

Dame Kathleen Ollerenshaw
(1912 – 2014)

We can add more magic to the 4 x 4 most perfect magic square. Our handout dedicated to Martin Gardner at the "Gathering for Gardner V", April 2002, Atlanta, Georgia was entitled "Five Card Study" and follows.

FIVE CARD STUDY
A Magic Divination

by Jeremiah Farrell

In his third collection of Scientific American columns, Martin Gardner describes the familiar trick of using five cards to divine a selected number that is based on the binary number system [G66]. Most mathemagicians are quite aware of this old chestnut. It has appeared on countless cereal boxes and in virtually all magic kits that have been sold for perhaps the last 100 years. To the young magician, however, it remains a fine introduction to both magic and mathematics and can still be highly recommended.

Our Five Card Study is a new effect, also based on five cards, that will confuse even the most sophisticated mathemagician. The five cards we have in mind are two-sided and a listing appears at the end of this article.

The effect: The magician shows the subject the road-map wheel with the 16 nodes, or stations, and the five colored routes between them. He explains its use with an example. "Suppose we decide to travel the red, green and yellow lines and choose to start at Station 3. We could go red to Station 14, green to Station 0 and, finally, end at Station 11 by traveling yellow."

After the subject understands the wheel, he is shown the five colored cards with the numbers on them. "All 16 numbers are on one side or the other of each of these colored cards," notes the magician, "and you may turn the five cards so that any combination of numbers is showing."

After the subject is satisfied with his placement of the five cards, he is asked to secretly jot down one of the 16 stations (0-15). The magician has previously written a prediction on a slip of paper.

Privately, the subject notes on which of the five colored cards his number appears, and, using these colors as routes (traveling the black line if his number appears on the white card) he travels from his station number on the road-map wheel. When he has completed his route, he informs the magician "I have arrived."

Even though the magician does not know the subject's start, routing, or end, he now directs the subject to continue traveling by calling out certain colors. It is found that the journey always ends at the magician's predicted station.

Another effect: The magician displays the 4x4 magic square that uses the numbers 0,1,2,. . ., 15 and marvels at the many ways the magic sum 30 appears on the board.

"Mathematicians call such a fecund magic square 'Most-Perfect' and the ancients considered such squares to be endowed with mystical powers," he says. The magician claims to have studied the powers of the square and proceeds to demonstrate.

"Choose one of the 16 numbers – do not tell me which one it is of course. I now show you five colored cards with the numbers of the square printed on one side or the other of each of them."

He holds each card in turn up to the subject's face, deliberately showing both sides, and places the cards down in front of the subject.

The magician allows the subject to turn over any of the five cards he cares to and adds, "I am going to ask you five simple yes-no questions and to make it harder on me, I want you to secretly choose to be either "convivial" and always tell the truth, or, to be "contrary" and always lie. That is, to tell five straight truths or five straight lies to the questions."

The questions are all of the form "Is your number here?" for each of the five cards and the "yes" responses are put to one side.

The magician glances at the magic square grid and quickly and correctly names the subject's number. (You may wish to have the subject write his number down earlier for verification.)

The method: No matter how the five colored cards are turned there will always be exactly one number of the 16 that either appears on all five cards or fails to appear on all five cards. This is called the "forced" number. Let us suppose as an example that the magician chooses 2 to be the forced number.

Turn the cards so that 2 appears on each one. Suppose the subject selects the number 6, and decides to lie. He will say his number appears on the red, green and blue cards. On the magic square, the magician mentally starts at 2 (the force), crosses the red edge to 15, the blue edge to 8, and the green edge to 6 the chosen number. If at any time the edges of the square are reached, the magician jumps to the other side of the square for blue or red yeses. For example, for the chosen 6, the magician could have started at the forced 2, go red to 15, then green to 1 and blue jump across to 6 as before. If the subject had decided to tell the truth instead of lying, he would have said yes to the yellow and white cards. Starting at 2 as before, the magician crosses yellow to 9 and a yes for white will always mean to make a (unique) diagonal hop - here to 6. (He could have started from 2, diagonally hopped to 13 and then crossed yellow to 6.) The road-map wheel works in a similar manner (recalling that a white card yes means travel the black line).

When the magician was holding the five cards up to the subject, he was really identifying his forced number and simply laid the cards down accordingly. If the magician had chosen 2 as the force, and the subject decided later to turn the blue and yellow cards over, the magician merely changes the force to 14 – a blue, yellow move from 2.

Follow-up trick: After performing the above effect the magician scoops up the yes responses (or the noes) and secretly turns them over. This changes the force to the subject's initial choice. In our example, turning yellow and white makes the force 6.

Show the subject the road-map wheel with the 16 nodes and the colored routes between them. Explain its use if this has not already been done. Have the subject choose another number and to note which of the cards his number is on. Ask him to travel from his new number on the colored routes he has selected. He will, much to his amazement, land on his first choice – 6 in our example.

The road-map wheel can be used to easily force a specific number on the entire audience by placing the five cards appropriately on an overhead screen (or on a TV monitor).

Either the magic square or the wheel can be regarded as a two-dimensional depiction of a five dimensional hypercube. Neither is a complete graph of the 5-cube since this would be overly confusing in two dimensions. Instead the 32 nodes are reduced by half by regarding each of the 16 numbers as being listed twice on the 32 nodes – once for convivial and once for contrary. This also reduces by half the number of other parts of the 5-cube. The reader will be able to find the 16 nodes, 40 lines, 40 faces, 20 cubes, and 5 tesseracts on the wheel by following the various colors. For a more complete discussion of hypercubes see [G75]. For another magic trick on the 4-cube, or tesseract, see [F02]. Most-Perfect magic squares are discussed in [OB98] and [P02].

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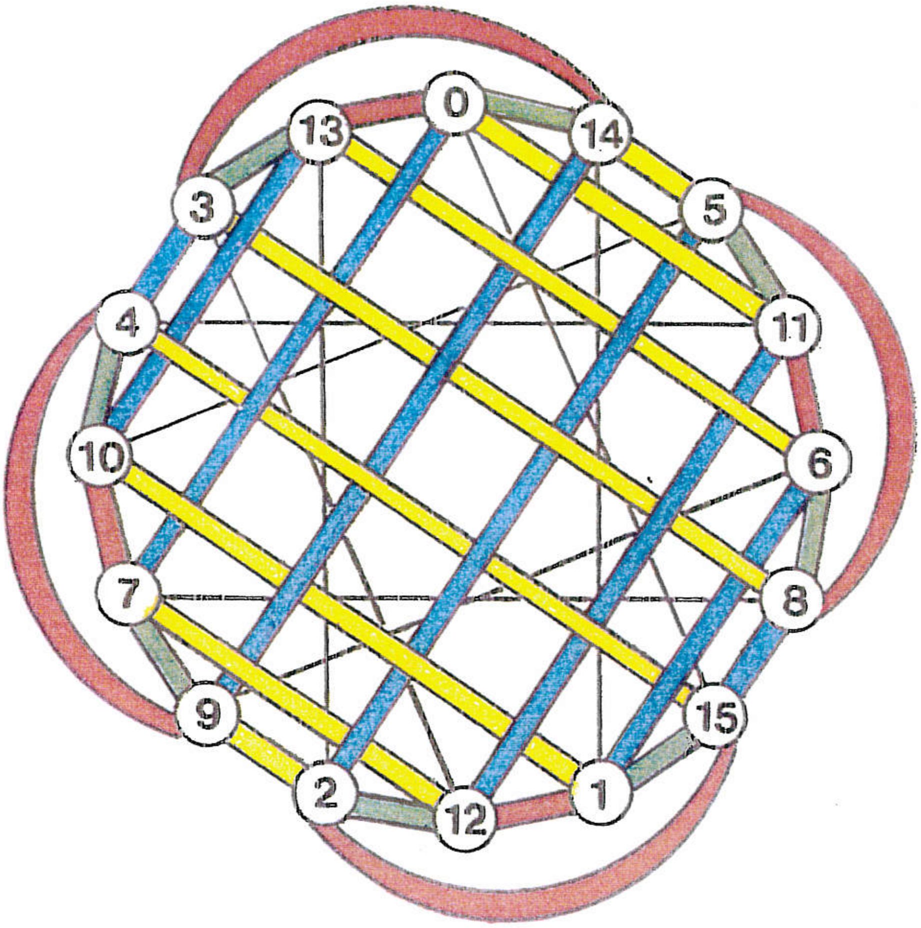
[F02] Jeremiah Farrell. "Cubist Magic". Puzzlers' Tribute. Edited by David Wolfe and Tom Rodgers. A K Peters, 2002.

[G66] Martin Gardner. New Mathematical Diversions from Scientific American. Simon and Schuster, 1966.

[G75] Martin Gardner. Mathematical Carnival. Knopf, 1975.

[OB98] Kathleen Ollerenshaw and David S. Bree. Most-Perfect Pandiagonal Magic Squares. Their construction and enumeration. The Institute of Mathematics and its Applications, 1998.

[P02] Clifford A. Pickover. The Zen of Magic Squares, Circles and Stars. Princeton Univ. Press, 2002.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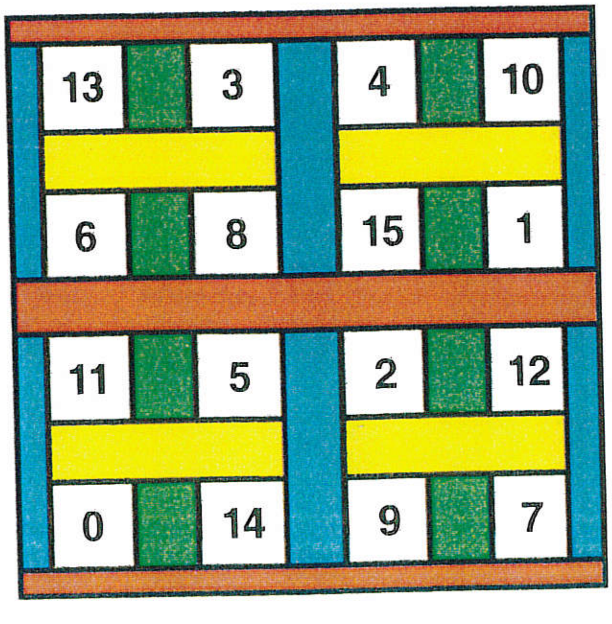
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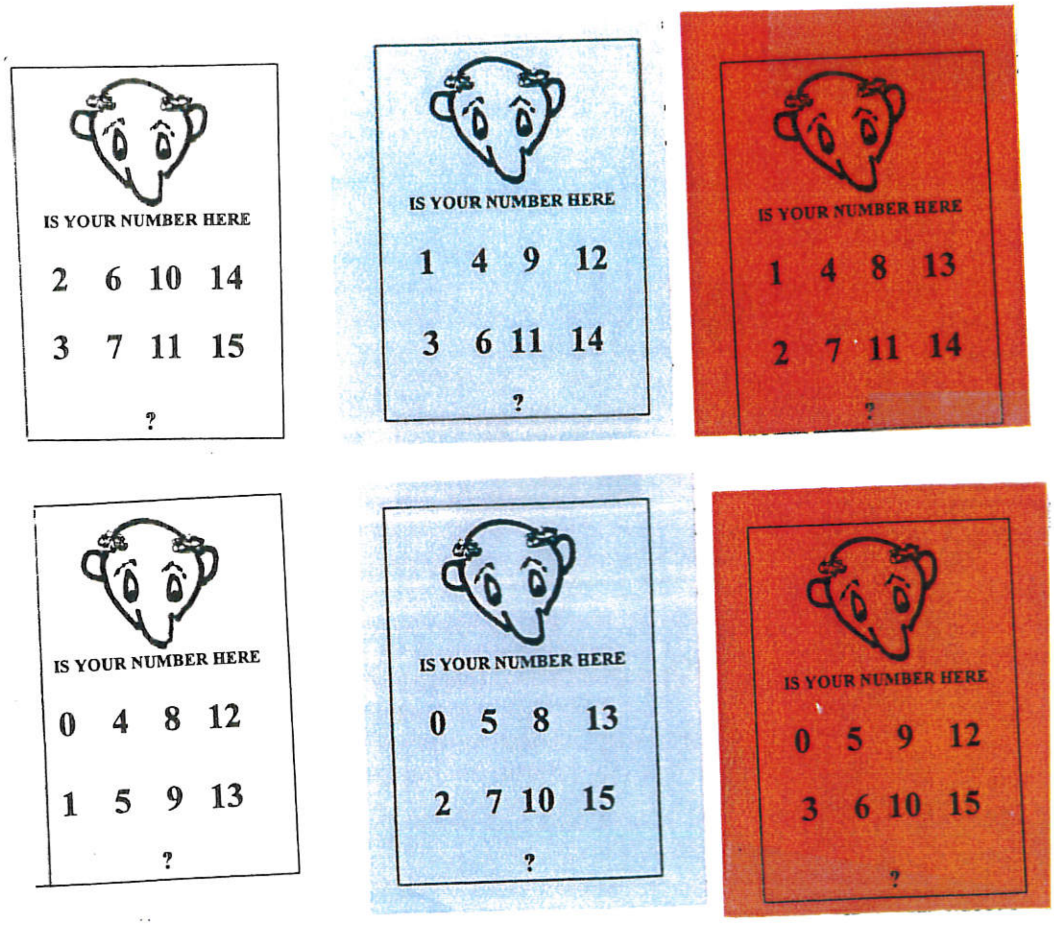
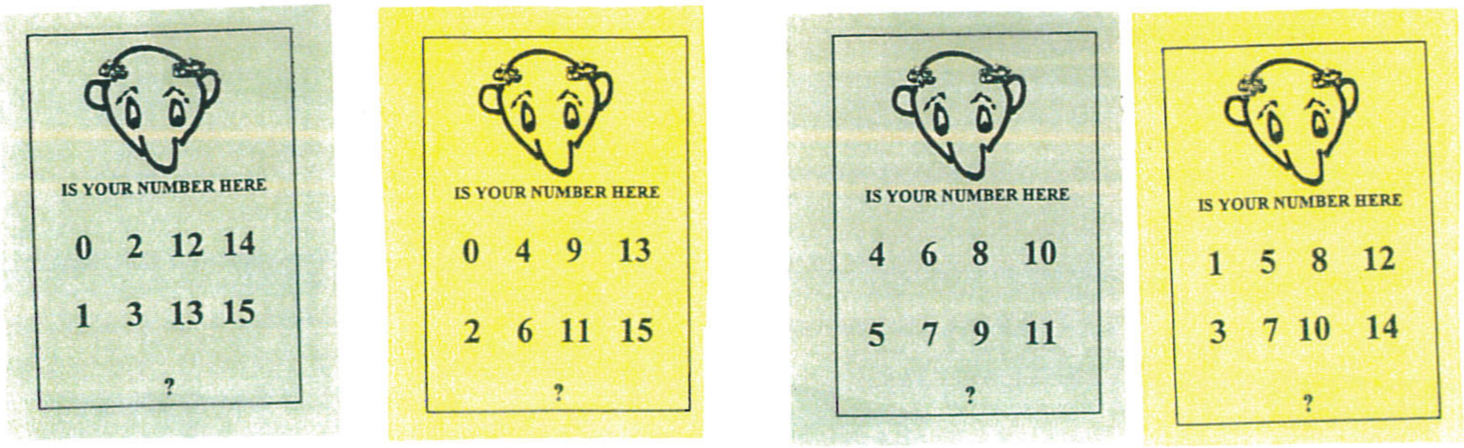


THE FIVE COLORED, DOUBLE SIDED CARDS



Another magic example is "Acey-Deucey".

ACEY – DEUCEY

The effect: The magician shows the subject the road-map wheel with the eight nodes and the colored routes between them. He explains its use with an example. "Suppose we decide to travel the red, green and yellow lines and choose to start at the ace of clubs. We could go red to the two of hearts, green to the two of spades and, finally, end on the ace of hearts by traveling yellow."

After the subject understands the wheel, he is shown the four colored cards with the aces and deuces on them. "All eight cards are on one side or the other of each of these colored cards," notes the magician, "and you may turn the four cards so that any combination of playing cards is showing."

After the subject is satisfied with his placement of the four cards, he is given a deck consisting of the four aces and four deuces which he shuffles thoroughly. He deals a card face-down to the magician and also to himself. Both look at their dealt cards.

Privately, the subject notes on which of the four colored cards his dealt card appears, and, using these colors as routes he travels from his dealt card on the road-map wheel. When he has completed his route, he informs the magician "I have arrived."

Even though the magician does not know the subject's start, routing, or end, he now directs the subject to continue traveling by calling out certain colors. It is found that the journey always ends on the magician's freely dealt card.

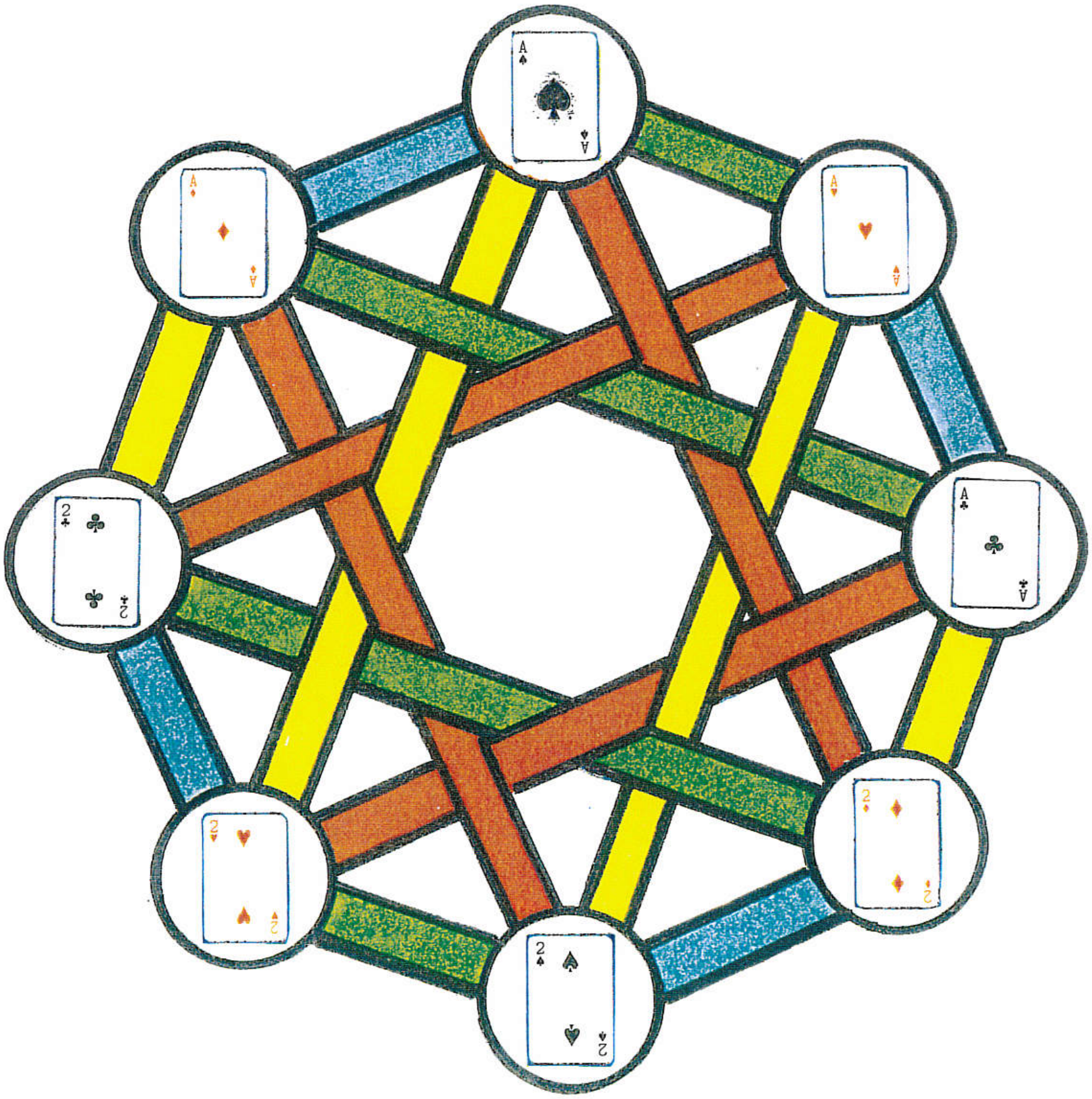
The method: No matter how the four colored cards are turned, there will always be exactly one card among the eight aces and deuces that either appears on all four cards or fails to appear on all four cards. This is called the "forced" card. The subject will <u>always</u> end on this card if he follows the directions correctly. Knowing this, the magician can easily find a route from the forced card to his own dealt card.

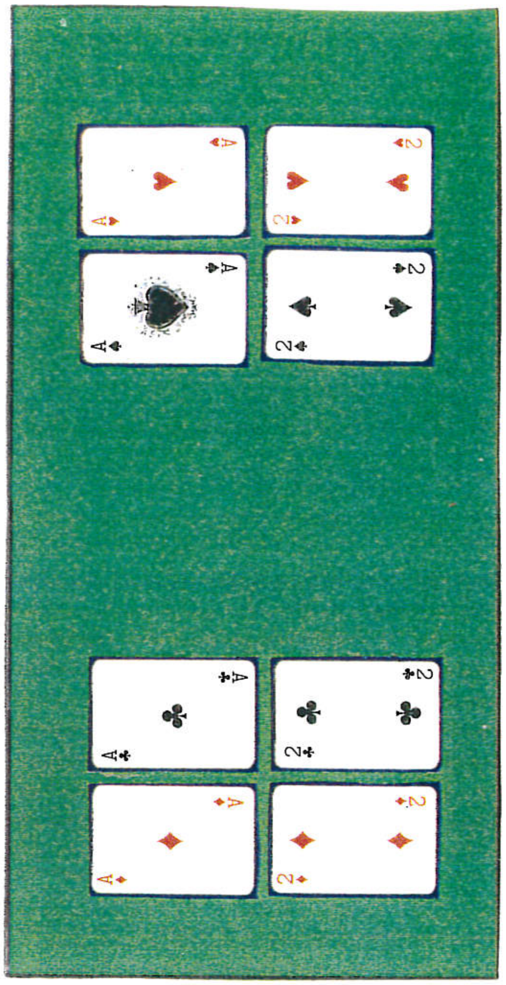
To facilitate locating the forced card the magician should note where the ace of spades, with its oversized pip, appears on the four cards. Trace those colors from the ace of spades on the wheel to find the force. For instance, suppose the ace of spades appears on the red and yellow cards. This will mean that the forced card is the ace of clubs. If the magician is dealt, say, the two of spades, he will direct the subject to travel the yellow and blue to arrive at the magician's card.

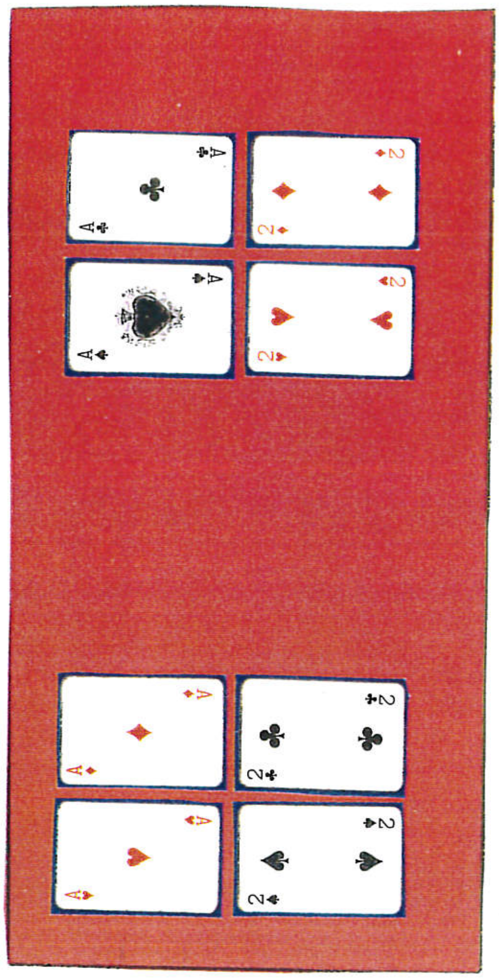
A variation: Mentally note the forced card and ask the subject to draw a card from the eight card deck. Allow him to secretly choose to be "convivial" and always tell the truth, or, to be "contrary" and always lie.

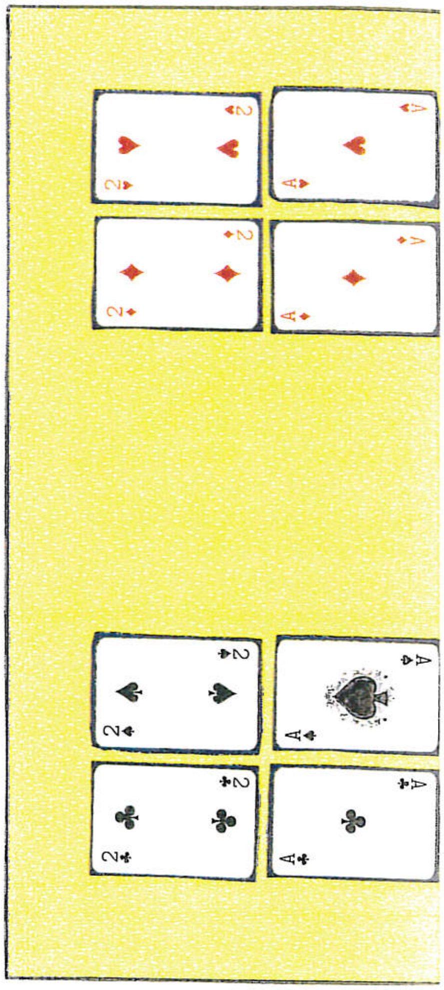
Then ask him to tell you which of the colored cards his card is on. Trace his yes responses from the forced card and, whether or not he was lying, you will always end on his card.

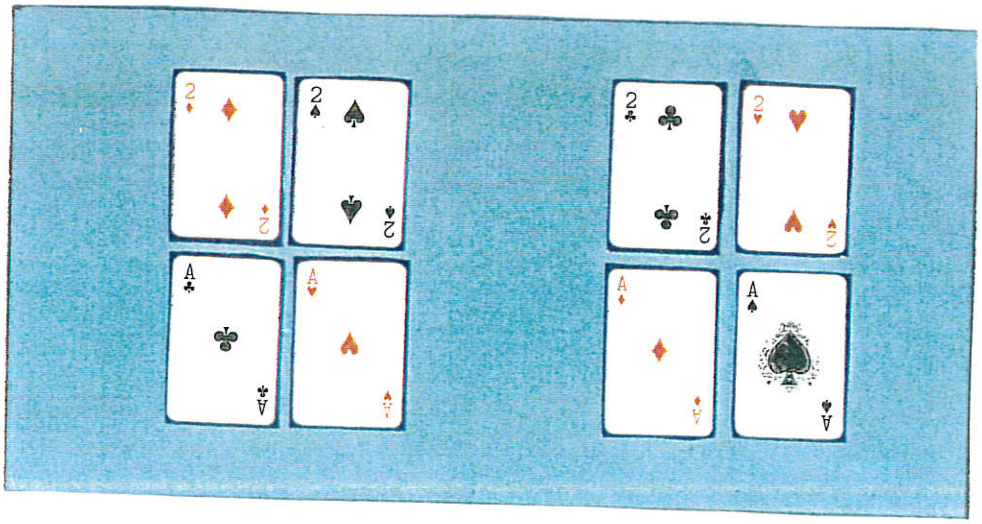
Another variation: The road-map wheel can be used to force the same card on everyone in an audience. Explain how to trace the colored routes and arrange the four colored cards so that a card of your choice is to be forced. Everyone in the audience, after choosing one of the eight cards, will travel to that forced card.











This square is a magic Acey-Deucey on the diagonals. The main diagonals are four spades and four diamonds. The half diagonals are four hearts and four clubs. The 1-3 diagonals are the black aces, the red aces, the black twos and the red twos.

AS	2H	2C	AD
AH	2S	2D	AC
2C	AD	AS	2H
2D	AC	AH	2S

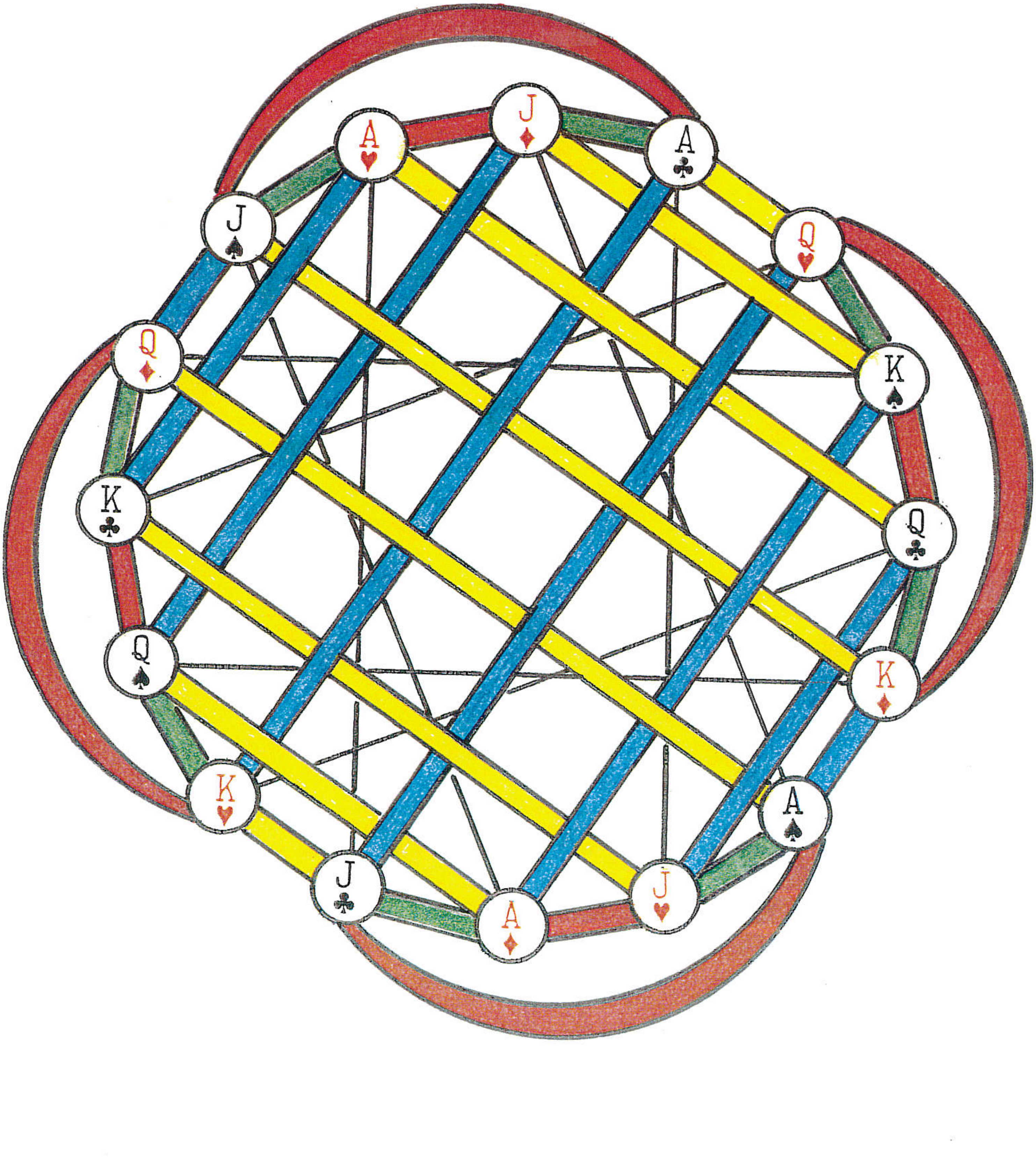
R G R G R

B Y B Y B

Fold R (Red) and B (Blue) into a torus, i.e., top to bottom and right to left. G and Y are Green and Yellow. The color markers need not be printed on the grid as they can easily be memorized. This square then can be used to magically find a spectator's choice by knowing only his color choices on the questions.

It is also possible for the audience to choose four entries, one from every column and row and the magician will be able to identify any one of the four from the other three by simply observing balance.

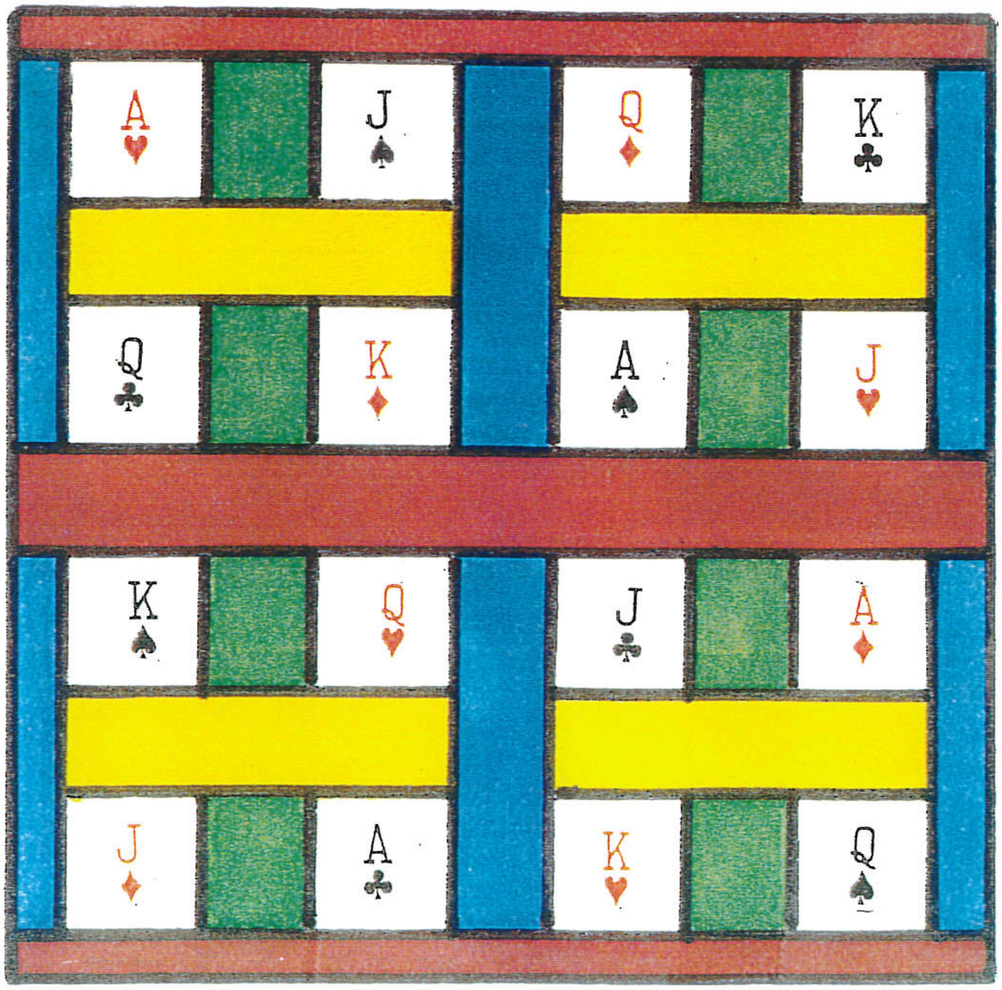
Magicians are no doubt familiar with the old puzzle of placing the 16 face cards of a deck into a 4 x 4 magic square. An early reference is Jacques Ozanam's 1694 edition *Récréations Mathématique et Physiques*. One example with its magic wheel follows.

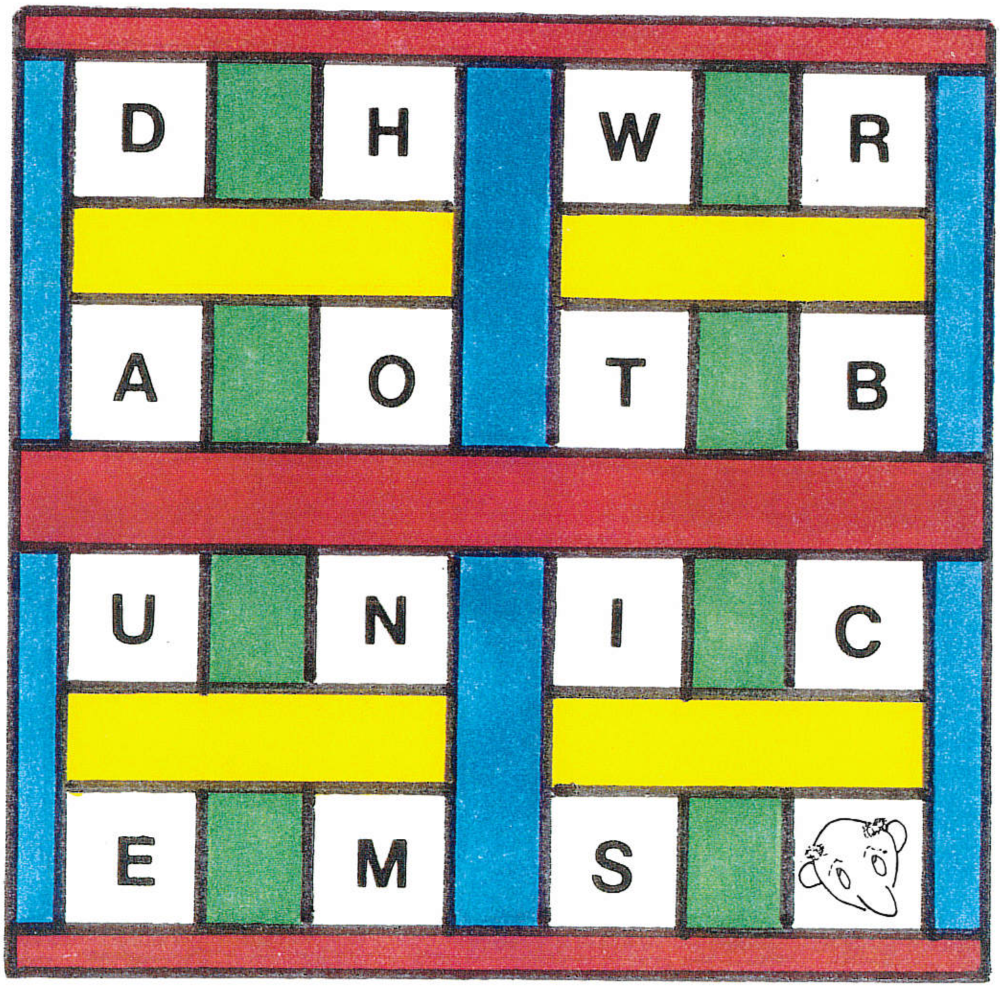


Notice the four corners of the square use each denomination and each suit exactly once and then both the denominations and suit are knight 4-hops through the square. The same effects can be accomplished here as were done in "Five Card Study" using the colored cards:

	Front	Back
RED	AS KH QH JS AC KC QC JD	AH KS QS JH AD KD QD JC
BLUE	AS KC QS JC AH KD QH JD	AC KS QC JS AD KH QD JH
GREEN	Ace or Jack	King or Queen
YELLOW	AS KS QC JC AH KH QD JD	AC KC QS JS AD KD QH JC
BLACK LINE	Any Red Card	Any Black Card

When the five cards are placed front side up the "forced" card is the Jack of Diamonds.





NORWICH BUMSTEAD DIVINES ALL!

THE NORWICH BUMSTEAD DIVINATION

The effect: The subject secretly chooses one of the fifteen letters in the name NORWICH BUMSTEAD. The magician then asks the subject to separate the five colored cards into two piles; one that contains the words that have his chosen letter in them and the other that contains those words that do not have his letter.

Even though the magician does not know which pile is which, he is quickly able to discern that subject's choice by a mere glance at the mystic NORWICH BUMSTEAD diagram.

The method: The mystic diagram can be thought of as a torus, or doughnut shape, by bending the half-red upper edge over to join the half-red lower edge, and then joining the half-blue ends to form the torus. This need not actually be done, of course, since it is easy to imagine that "A" is connected to the "B" via a blue edge, or that the "W" is connected through a red edge to the "S", and so on. With this proviso, every letter (and Norwich himself in the lower right-hand corner) is connected to exactly four other letters by crossing one of the four colors Red, Blue, Yellow or Green.

When the subject separates the five colored cards, the magician simply notes the colors in <u>either</u> pile. He mentally traces the colors in that pile, starting from the Norwich square in the lower right. If the white card is in his chosen pile, the magician makes a (unique) diagonal jump over one square to account for it. For example, suppose Red, Green and White are in one pile. From Norwich the magician crosses Red to "R", Green to "W" and then does the White diagonal hop to land on "U", the chosen letter. Had he traced the other pile instead, i.e., Blue and Yellow, he could have gone Yellow to "C' and then Blue to "U" reaching the same spot. It is worth noting that the colors in any pile may be traced in any order; they will always locate the same letter.