A short story in alphametics

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The following independent alphametics form a short story about TeX and other programs and books by Donald E. Knuth. The program for TeX is available as Volume B of *Computers & Typesetting* (or *C&T* for short); see `www-cs-faculty.stanford.edu/~knuth/abcde.html`. Volume A is the user manual named *The TeXbook* and three more volumes describe the program Metafont and the Computer Modern fonts. The programs have a big user base and the users join together, for example, in the international TeX Users Group (TUG); see `www.tug.org`.

$$\mathtt{ONCE} + \mathtt{D} + \mathtt{KNUTH} = \mathtt{CODED} + \mathtt{TEX}$$

$$\mathtt{DEK} + \mathtt{WROTE} = \mathtt{THE} + \mathtt{TEX} + \mathtt{BOOK} + \mathtt{TOO} \qquad \text{and} \quad 0 \le \mathtt{O} < \mathtt{K}$$

$$\mathtt{AND} + \mathtt{DEK} + \mathtt{MADE} + \mathtt{MORE} = \mathtt{OK} + \mathtt{READ} + \mathtt{CANDT}$$

$$\mathtt{TUG} + \mathtt{MEMBER} = \mathtt{TEX} + \mathtt{USER} + \mathtt{MEETS} + \mathtt{MORE} + \mathtt{TEX} + \mathtt{USERS}$$

The second alphametic has two solutions, all others are pure, i.e., they have a unique solution. The additional condition for the second alphametic makes it pure too and the condition is stated in a way that helps if the alphametic is solved by hand. To solve the third alphametic by hand use the following additional information $\mathtt{E} + \mathtt{R} + \mathtt{R} = \mathtt{T}$ if you get stuck.

The solution

First, the four solutions to the alphametics:

O = 2, N = 9, C = 8, E = 6, D = 1, K = 7, U = 5, T = 3, H = 4, X = 0.

D = 6, E = 2, K = 4, W = 1, R = 0, O = 3, T = 5, H = 8, X = 7, B = 9.

A = 3, N = 6, D = 5, E = 4, K = 0, M = 7, O = 9, R = 2, C = 1, T = 8.

T = 5, U = 9, G = 8, M = 1, E = 2, B = 0, R = 4, X = 6, S = 7, O = 3.

And here are the summations and the carries:

```
     ONCE        2986         2986
+       D   +       1   +        1          82161
+   KNUTH   +   79534   +    79534   +        360
   ------      ------         1111              1
    CODED       82161       ------         ------
+     TEX   +     360        82521          82521
```

```
      DEK         624                         582
+   WROTE   +   10352                  +      527
   ------      ------          624     +     9334
      THE         582   +    10352     +      533
+     TEX   +     527                        1111
+    BOOK   +    9334       ------         ------
+     TOO   +     533        10976          10976
```

```
      AND         365          365
+     DEK   +     540          540    +        90
+    MADE   +    7354   +     7354    +      2435
+    MORE   +    7924   +     7924    +     13658
   ------      ------          211            111
       OK          90       ------         ------
+    READ   +    2435        16183          16183
+   CANDT   +   13658
```

```
      TUG         598                         526
+  MEMBER   +  121024                  +     9724
   ------      ------                  +    12257
      TEX         526                  +     1342
+    USER   +    9724          598     +      526
+   MEETS   +   12257   +   121024     +    97247
+    MORE   +    1342           11           2223
+     TEX   +     526       ------         ------
+   USERS   +   97247       121622         121622
```

Manual calculations

In the first alphametic one of N, E, U, H, X is 0 as words cannot start with a zero. Let's rewrite the equation as

$$\mathtt{E} + \mathtt{D} + \mathtt{H} \equiv \mathtt{D} + \mathtt{X} \pmod{10} \tag{1}$$

$$\mathtt{C} + \mathtt{T} + a_1 \equiv 2\mathtt{E} + b_1 \pmod{10} \tag{2}$$

$$\mathtt{N} + \mathtt{U} + a_2 \equiv \mathtt{D} + \mathtt{T} + b_2 \pmod{10} \tag{3}$$

$$\mathtt{O} + \mathtt{N} + a_3 \equiv \mathtt{O} + b_3 \pmod{10} \tag{4}$$

$$\mathtt{K} + a_4 \equiv \mathtt{C} + b_4 \pmod{10} \tag{5}$$

where a_i and b_i, $1 \leq i \leq 4$, are the carries from column i, i.e., the column to the right. As the sum of two different digits is at most 17 all carries except a_1 are either 0 or 1; a_1 might be 2.

Now (5) shows that $a_4 \neq b_4$ as $\mathtt{K} \neq \mathtt{C}$; either $\mathtt{O} + \mathtt{N} + a_3$ or $\mathtt{O} + b_3$ of (4) must produce a carry. If $a_4 = 0$ but $b_4 = 1$ then $\mathtt{O} = 9$ and $b_3 = 1$ as well as $\mathtt{N} = 0$ and $a_3 = 0$. This means $9 + 0 + 0 \equiv 9 + 1 \pmod{10}$, a contradiction. Thus $a_4 = 1$, $b_4 = 0$, and $\mathtt{K} + 1 = \mathtt{C}$; moreover $\mathtt{C} > 1$ as $\mathtt{K} \neq 0$.

Subtracting O on both sides of (4) states $\mathtt{N} + a_3 \equiv b_3 \pmod{10}$. As a_3 and b_3 are either 0 or 1 one of three cases must be true: $(\mathtt{N}; a_3, b_3)$ is either $(9; 1, 0)$ or $(1; 0, 1)$ or $\mathtt{N} = 0$ and $a_3 = b_3$. The last case is impossible as (4) becomes $\mathtt{O} + 0 + a_3 \equiv \mathtt{O} + b_3 \pmod{10}$ and that means $a_4 = b_4$, which is not allowed. Similar $\mathtt{N} = 1$ gives $\mathtt{O} + 1 + 0 \equiv \mathtt{O} + 1 \pmod{10}$ and this implies $a_4 = b_4$. Therefore, $a_3 = 1$, $b_3 = 0$, and $\mathtt{N} = 9$.

Next, we look at (1): $\mathtt{E}+\mathtt{D}+\mathtt{H} \equiv \mathtt{D}+\mathtt{X} \pmod{10}$ or $\mathtt{E}+\mathtt{H} \equiv \mathtt{X} \pmod{10}$. This means that neither E nor H can be 0 elsewise the other must be equal to X. So either U or X must be 0. If $\mathtt{U} = 0$ then (3) becomes $9 + 0 + a_2 \equiv \mathtt{D} + \mathtt{T} + b_2 \pmod{10}$. The finding $a_3 = 1$ requires that $a_2 = 1$ and thus $0 \equiv \mathtt{D} + \mathtt{T} + b_2 \pmod{10}$. But as $b_3 = 0$ the sum $\mathtt{D} + \mathtt{T} + b_2$ must equal 0; a contradiction because $\mathtt{D} + \mathtt{T} > 0$. Thus $\mathtt{X} = 0$.

Back to (1): As $\mathtt{D} + \mathtt{X} = \mathtt{D} + 0 < 10$ we know that $b_1 = 0$. And $a_1 = 1$ as $\mathtt{E} + \mathtt{H} \equiv 0 \pmod{10}$ or $\mathtt{E} + \mathtt{H} = 10$ so that $10 < \mathtt{E} + \mathtt{D} + \mathtt{H} < 20$. There are only three sets that qualify for $\{\mathtt{E}, \mathtt{H}\}$: $\{2, 8\}$, $\{3, 7\}$, $\{4, 6\}$.

Congruence (2) becomes $\mathtt{C} + \mathtt{T} \equiv 2\mathtt{E} - 1 \pmod{10}$. The six possible values for E help to find candidates for $(\mathtt{C}, \mathtt{T})$. The values 0, 9, and the associated H are excluded for C and T and we know $\mathtt{C} > 1$ and neither E nor T can be $\mathtt{C} - 1 = \mathtt{K}$. For $(\mathtt{E}, 2\mathtt{E} - 1; \mathtt{C}, \mathtt{T})$ we find $(2, 3; 6, 7)$, $(4, 7; 2, 5)$, $(6, 11; 3, 8)$, $(6, 11; 8, 3)$, $(7, 13; 5, 8)$, $(8, 15; 4, 1)$.

Only six assignments for $(\mathtt{N}, \mathtt{C}, \mathtt{E}, \mathtt{K}, \mathtt{T}, \mathtt{H}, \mathtt{X})$ might lead to a valid solution: $(9, 2, 4, 1, 5, 6, 0)$, $(9, 3, 6, 2, 8, 4, 0)$, $(9, 4, 8, 3, 1, 2, 0)$, $(9, 5, 7, 4, 8, 3, 0)$, $(9, 6, 2, 5, 7, 8, 0)$, $(9, 8, 6, 7, 3, 4, 0)$. Now three letters are still missing: U, D, O. Their values in the six partial assignments must form the respective set $\{3, 7, 8\}$, $\{1, 5, 7\}$, $\{5, 6, 7\}$, $\{1, 2, 6\}$, $\{1, 3, 4\}$, or $\{1, 2, 5\}$. As $a_3 = 1$ and $b_3 = 0$ we get from (3) an equation $\mathtt{U} + a_2 = \mathtt{D} + \mathtt{T} + b_2 + 1$. Based on the sets of the missing letters the values of T, C, and E from the corresponding partial assignment are used to calculate the values a_2 and b_2 from (2): $\mathtt{C} + \mathtt{T} > 8 \iff a_2 = 1$, $\mathtt{E} > 4 \iff b_2 = 1$. The sum of $\mathtt{T} + b_2 + 1 - a_2$ and D must equal U. There is only one solution: $(\mathtt{U}, \mathtt{D}, \mathtt{O}) = (5, 1, 2)$ for $(\mathtt{N}, \mathtt{C}, \mathtt{E}, \mathtt{K}, \mathtt{T}, \mathtt{H}, \mathtt{X}) = (9, 8, 6, 7, 3, 4, 0)$.

The alphametic has a unique solution: $(\mathtt{O}, \mathtt{N}, \mathtt{C}, \mathtt{E}, \mathtt{D}, \mathtt{K}, \mathtt{U}, \mathtt{T}, \mathtt{H}, \mathtt{X}) = (2, 9, 8, 6, 1, 7, 5, 3, 4, 0)$.

In the second alphametic one of E, K, R, O, H, X must be 0. This time we have

$$\mathtt{K} + \mathtt{E} \equiv \mathtt{E} + \mathtt{X} + \mathtt{K} + \mathtt{O} \pmod{10} \tag{6}$$

$$\mathtt{E} + \mathtt{T} + a_1 \equiv \mathtt{H} + \mathtt{E} + 2\mathtt{O} + b_1 \pmod{10} \tag{7}$$

$$\mathtt{D} + \mathtt{O} + a_2 \equiv 3\mathtt{T} + \mathtt{O} + b_2 \pmod{10} \tag{8}$$

$$\mathtt{R} + a_3 \equiv \mathtt{B} + b_3 \pmod{10} \tag{9}$$

$$\mathtt{W} + a_4 \equiv b_4 \pmod{10} \tag{10}$$

$$\mathtt{O} < \mathtt{K} \tag{11}$$

where the a_i and b_i, $1 \leq i \leq 4$, are the carries.

The only possibility for (10) is $b_4 = 1$ (and thus $b_3 > 0$), $a_4 = 0$, and $\mathtt{W} = 1$.

In (6) K and in (6) and (7) E appear on both sides and after subtraction both disappear from the formulas, i.e., their values can be exchanged. (11) assures that the solution is unique, so $\mathtt{E} < \mathtt{O} < \mathtt{K}$. And (6) without E and K shows that $0 \equiv \mathtt{X}+\mathtt{O}$ or $\{\mathtt{X}, \mathtt{O}\} \in \{\{2, 8\}, \{3, 7\}, \{4, 6\}\}$. And now one of E, R, or H must be 0.

Formula (9) and $b_4 = 1$ give $\mathtt{B} \geq 7$ as $0 < b_3 \leq 3$. Note $\mathtt{R} \neq 1 = \mathtt{W}$ and $\mathtt{R} < 9$ if $a_3 = 1$ as $a_4 = 0$. So $(\mathtt{B}, b_3; \mathtt{R}, a_3)$ is one of $(7, 3; 0, 0)$, $(8, 3; 0, 1)$, $(8, 2; 0, 0)$, $(9, 3; 2, 0)$, $(9, 2; 0, 1)$, or $(9, 1; 0, 0)$.

Can R equal 2? Then $\mathtt{B} = 9$, $b_3 = 3$, and $a_3 = 0$. Thus $\mathtt{D} + \mathtt{O} + a_2 < 10$. As now $\mathtt{D} > 2$ we must have $\mathtt{O} < 7 - a_2$, i.e., $\mathtt{O} \in \{2, 3, 4, 6\}$. The right hand side of (7) is therefore at most $(8 + 0) + 2 \cdot 6 + 2 = 22$ and thus $b_2 \leq 2$ and T must be 8, $\mathtt{O} = 6$, and $b_2 = 2$ to make $b_3 = 3$; then $\mathtt{X} = 4$ and $\mathtt{K} = 7$. As $a_3 = 0$ (8) gives $\mathtt{D} = 3$ and $a_2 = 0$. This means by (7) $\mathtt{E} = 0$ and H gets the remaining digit, the 5. Therefore $b_1 = 1$ and $b_2 = 1$ as $\mathtt{H} + \mathtt{E} + 2\mathtt{O} + b_1 = 18$; contradiction because b_2 was found to be 2. Thus, $\mathtt{R} = 0$.

As $\mathtt{R} = 0$ E must be at least 2 and thus $\mathtt{O} > 2$. Moreover $\mathtt{B} = 9$ and $\mathtt{O} = 8$ violates (11) as there is no value for K. And $\mathtt{B} = 7$ and $\mathtt{O} = 8$ implies $\mathtt{K} = 9$ so that $\mathtt{T} < 7$ and the right hand side of (8) is at most $3 \cdot 6 + 8 + 3 = 29$. Thus $b_3 < 3$; a contradiction as $(\mathtt{B}; b_3)$ must be $(7; 3)$. Therefore $2 < \mathtt{O} \neq 8$. Thus, $\{\mathtt{X}, \mathtt{O}\} \in \{\{3, 7\}, \{4, 6\}\}$.

Let's summarize what we have found so far. The tuple $(\mathtt{W}, \mathtt{R}, \mathtt{O}, \mathtt{X}, \mathtt{B})$ is one of ten cases: $(1, 0, 4, 6, 7)$, $(1, 0, 6, 4, 7)$ with $(b_3, a_3) = (3, 0)$; or $(1, 0, 3, 7, 8)$, $(1, 0, 4, 6, 8)$, $(1, 0, 6, 4, 8)$, $(1, 0, 7, 3, 8)$ with two cases for (b_3, a_3): either $(2, 0)$ or $(3, 1)$; or $(1, 0, 3, 7, 9)$, $(1, 0, 4, 6, 9)$, $(1, 0, 6, 4, 9)$, $(1, 0, 7, 3, 9)$ and $(b_3, a_3) \in \{(1, 0), (2, 1)\}$.

For the four possible values of O (7) is used to find acceptable (H, T) pairs. As $b_1 - a_1 = 1$ the formula can be shortened to $2\mathtt{O} + 1 + \mathtt{H} \equiv \mathtt{T} \pmod{10}$. The digits of the set $\{0, 1, \mathtt{O}, \mathtt{X}\}$ are ignored for T and H. And the pairs must allow to assign a value less than O to E as well larger values to B and K. A little computation gives seven pairs: $(\mathtt{O}, 2\mathtt{O} + 1; \mathtt{H}, \mathtt{T})$ is one of $(3, 7; 8, 5)$, $(3, 7; 9, 6)$, $(4, 9; 8, 7)$, $(4, 9; 9, 8)$, $(6, 13; 2, 5)$, $(6, 13; 5, 8)$, or $(6, 13; 9, 2)$.

Now (8) is applied to the five values of T together with all possible values for $b_2 \in \{1, 2\}$ and $a_2 \in \{0, 1\}$ to get D and combine it with O and H. Finally, we compute a_3 and b_3. In total 20 cases have to be looked at but only a few valid tuples are found. $(\mathtt{T}, \mathtt{D}, \mathtt{O}, \mathtt{H}; b_3, a_3)$ is one of $(5, 6, 3, 8; 1, 0)$, $(5, 6, 3, 8; 2, 1)$, $(5, 7, 6, 2; 2, 1)$, $(6, 8, 3, 9; 2, 1)$, $(7, 2, 4, 8; 2, 0)$, $(7, 3, 4, 8; 2, 0)$, $(8, 5, 4, 9; 2, 0)$, $(8, 5, 4, 9; 3, 1)$.

These eight tuples are combined with the ten from above using the letter $\texttt{O}$ and only if the values of a_3 and b_3 agree. Two cases remain: $(\texttt{D}, \texttt{W}, \texttt{R}, \texttt{O}, \texttt{T}, \texttt{H}, \texttt{X}, \texttt{B})$ must be one of $(6, 1, 0, 3, 5, 8, 7, 9)$ or $(7, 1, 0, 6, 5, 2, 4, 9)$. In the first case $(\texttt{E}, \texttt{K}) = (2, 4)$ in the second $(3, 8)$.

For the second case the alphametic becomes $738 + 10653 = 11391$ and $523 + 534 + 9668 + 566 = 11291$; so this assignment is not a solution. The unique solution of the problem is: $(\texttt{D}, \texttt{E}, \texttt{K}, \texttt{W}, \texttt{R}, \texttt{O}, \texttt{T}, \texttt{H}, \texttt{X}, \texttt{B}) = (6, 2, 4, 1, 0, 3, 5, 8, 7, 9)$.

In the third alphametic only four letters can be 0: $\texttt{N}$, $\texttt{E}$, $\texttt{K}$, or $\texttt{T}$. The equations create the following formulas:

$$\texttt{D} + \texttt{K} + 2\texttt{E} \equiv \texttt{K} + \texttt{D} + \texttt{T} \pmod{10} \tag{12}$$

$$\texttt{N} + \texttt{E} + \texttt{D} + \texttt{R} + a_1 \equiv \texttt{O} + \texttt{A} + \texttt{D} + b_1 \pmod{10} \tag{13}$$

$$2\texttt{A} + \texttt{D} + \texttt{O} + a_2 \equiv \texttt{E} + \texttt{N} + b_2 \pmod{10} \tag{14}$$

$$2\texttt{M} + a_3 \equiv \texttt{R} + \texttt{A} + b_3 \pmod{10} \tag{15}$$

$$a_4 \equiv \texttt{C} + b_4 \pmod{10} \tag{16}$$

$$\texttt{E} + 2\texttt{R} = \texttt{T} \tag{17}$$

where the a_i and b_i, $1 \leq i \leq 4$, are the carries. Equation (17) is the hint from the problem statement to simplify the computation.

As $a_4 \leq 2$ by (16) $1 \leq \texttt{C} \leq 2$. $\texttt{C} = 2$ means $a_4 = 2$ and $b_4 = 0$, thus by (15) $\texttt{M} = 9$ and $a_3 \geq 2$ as well as $\texttt{R} + \texttt{A} + b_3 < 10$. As $a_3 < 4$ the left hand side of (15) is either 20 or 21 so $\texttt{R} + \texttt{A} + b_3$ must be 0 or 1 which is impossible as neither $\texttt{R}$ nor $\texttt{A}$ can be 0. Thus, $\texttt{C} = 1$.

Congruence (12) can be reduced to $2\texttt{E} \equiv \texttt{T} \pmod{10}$ which means that $\texttt{T}$ must be even and $\texttt{E} \neq 0$. Of course (17) states that $\texttt{E}$ must be even too. And it must be smaller than $\texttt{T}$. Thus there are only two cases for $(\texttt{E}, \texttt{T})$: $(2, 4)$ or $(4, 8)$. As $\texttt{R} > 1$ only the second case is valid: $\texttt{R} = 2$, $\texttt{E} = 4$, and $\texttt{T} = 8$. And as $2\texttt{E} = \texttt{T}$ we know $a_1 = b_1$.

If $b_4 = 1$ then $a_4 = 2$, $\texttt{M} = 9$, and $a_3 \geq 2$. Moreover $b_3 \leq 1$ and with (15) this means $\texttt{A} = 7$, $b_3 = 1$, $a_3 = 2$, $\texttt{N} \neq 0$ and thus $\texttt{K} = 0$ as it is the remaining letter of the above list. So $\{\texttt{N}, \texttt{D}, \texttt{O}\} = \{3, 5, 6\}$. Formula (13) states now $\texttt{N} \equiv \texttt{O} + 1 \pmod{10}$ so that $\texttt{N} = 6$, $\texttt{O} = 5$, and therefore $\texttt{D} = 3$. This leads to $763 + 340 + 9734 + 9524 = 20361$ and $50 + 2473 + 17638 = 20161$ which is not a solution of the alphametic. Thus $b_4 = 0$ and $a_4 = 1$.

If $\texttt{M} \leq 5$ then by (15) $10 \leq 2\texttt{M} + a_3 < 14$ and $2 + \texttt{A} + b_3 < 4$. A contradiction as $\texttt{A} \geq 3$. If $\texttt{M} = 6$ then a_3 must be 3 to reach 15. But as then $2\texttt{A} + \texttt{D} + \texttt{O} + a_2 \leq 6 + 9 + 8 + 3 = 26$ we have $a_3 \leq 2$; therefore $\texttt{M} \in \{7, 9\}$. As $\texttt{A} > 7$ implies $b_4 > 0$ we find $\texttt{A} \in \{3, 5, 6, 7\}$.

With all these values the left hand side of (14) is between $17 + a_2$ and $26 + a_2$, i.e., $a_3 \in \{1, 2\}$, and the right hand side is smaller than 20, so that $b_3 \in \{0, 1\}$. There are just three cases for (15) as $\texttt{M} = 9 \Rightarrow a_3 < 2$ and we find two solutions both with $\texttt{M} = 7$ and $\texttt{A} = 3$.

The four remaining letters $\texttt{N}$, $\texttt{D}$, $\texttt{K}$, and $\texttt{O}$ must be assigned to the set $\{0, 5, 6, 9\}$. Substituting the known values in (13) gives $\texttt{N} + 4 + \texttt{D} + 2 + a_1 \equiv \texttt{O} + 3 + \texttt{D} + b_1 \pmod{10}$ or $\texttt{N} + 3 \equiv \texttt{O} \pmod{10}$. Thus $\texttt{N} = 6$ and $\texttt{O} = 9$.

Next, $\texttt{K} = 0$ and as there is only one digit left, $\texttt{D} = 5$.

The equations have a unique solution: $(\texttt{A}, \texttt{N}, \texttt{D}, \texttt{E}, \texttt{K}, \texttt{M}, \texttt{O}, \texttt{R}, \texttt{C}, \texttt{T}) = (3, 6, 5, 4, 0, 7, 9, 2, 1, 8)$.

In the fourth alphametic all letters except $\mathtt{T}$, $\mathtt{U}$, and $\mathtt{M}$ can be 0. This time we have

$$\mathtt{G}+\mathtt{R} \equiv 2\mathtt{X}+\mathtt{R}+2\mathtt{S}+\mathtt{E} \pmod{10} \tag{18}$$

$$\mathtt{U}+\mathtt{E}+a_1 \equiv 3\mathtt{E}+2\mathtt{R}+\mathtt{T}+b_1 \pmod{10} \tag{19}$$

$$\mathtt{T}+\mathtt{B}+a_2 \equiv 2\mathtt{T}+2\mathtt{E}+\mathtt{S}+\mathtt{O}+b_2 \pmod{10} \tag{20}$$

$$\mathtt{M}+a_3 \equiv \mathtt{U}+\mathtt{E}+\mathtt{M}+\mathtt{S}+b_3 \pmod{10} \tag{21}$$

$$\mathtt{E}+a_4 \equiv \mathtt{M}+\mathtt{U}+b_4 \pmod{10} \tag{22}$$

$$\mathtt{M}+a_5 \equiv b_5 \pmod{10} \tag{23}$$

where the a_i and b_i, $1 \le i \le 5$, are the carries.

If $a_5 = 1$ then $\mathtt{E} = 9$ and $a_4 = 1$, but then $\mathtt{M}+a_3 < 10$ as $a_3 \in \{0,1\}$, so $a_4 = 0$; contradiction. Thus $a_5 = 0$. As $b_4 < 5$ the sum $\mathtt{M}+\mathtt{U}+b_4 \le \mathtt{M}+9+4 = \mathtt{M}+13$ has a carry b_5 of 1 or 2. But by (23) $\mathtt{M}$ equals also b_5 so both must be 1.

Of course, $a_i \in \{0,1\}$ for $1 \le i \le 3$ and $a_4 = a_5 = 0$. The ranges for the other carries except b_5 are limited by using the smallest or largest available digits in the congruences. The result is: $1 \le b_1 \le 4$, $0 \le b_2 \le 5$, $1 \le b_3 \le 5$, $0 \le b_4 \le 3$. To make $b_4 = 3$ we need $\mathtt{U}+\mathtt{E}+1+\mathtt{S}+b_3 \ge 30$ or $b_3 = 5$ and $\mathtt{U}+\mathtt{E}+\mathtt{S} = 24$, i.e., $\{\mathtt{U},\mathtt{E},\mathtt{S}\} = \{9,8,7\}$. But the maximum for b_3 is then 4 as by (20) $2\cdot 6+2\cdot 9+8+5+b_2 = 43+b_2 < 50$; so $0 \le b_4 \le 2$.

As $b_5 = 1$ the right hand side of (22) must create a carry; this limits the possibilities for $(b_4;\mathtt{U},\mathtt{E})$: $(0;9,0)$, $(1;8,0)$, $(2;7,0)$, $(2;9,2)$. The first case is invalid as the right hand side of (21) is larger than 10 if $\mathtt{U} = 9$ so that $b_4 > 0$.

If $\mathtt{E} = 0$ then by (22) $\mathtt{U}+b_4 = 9$ with three possibilities for b_4: 0, 1, 2. If $b_4 = 0$ then $\mathtt{U} = 9$ so that the right hand side of (21) gets larger than 10, i.e., $b_4 > 0$; a contradiction. Similar $b_4 = 2$ means $\mathtt{U} = 7$ and the right hand side of (21) must be 21 or 22. Even with $\mathtt{S} = 9$ b_3 must be 4 or 5 which is impossible as the right hand side of (20) is at most $31+b_2$. Therefore $b_4 = 1$ and $\mathtt{U} = 8$. If $a_3 = 0$ then by (21) $0 \equiv 8+\mathtt{S}+b_3 \pmod{10}$ so that $\mathtt{S} = 2$ and $b_3 = 0$; but the right hand side of (20) makes $b_3 > 0$. If $a_3 = 1$ then either $\mathtt{S} = 3$ and $b_3 = 0$, which leads to a similar contradiction as before, or $\mathtt{S} = 2$ and $b_3 = 1$. Now by (18) $\mathtt{G} \equiv 2\mathtt{X}+4 \pmod{10}$ so $\mathtt{G}$ is even: $\mathtt{G} \in \{4,6\}$ as $\mathtt{E} = 0$, $\mathtt{S} = 2$, $\mathtt{U} = 8$ in this case. In the first case $\mathtt{X} = 0$ or $\mathtt{X} = 5$, in the second $\mathtt{X} = 1$ or $\mathtt{X} = 6$. Only the pair $(\mathtt{G},\mathtt{X}) = (4,5)$ is possible. So we have $\{\mathtt{R},\mathtt{T},\mathtt{B},\mathtt{O}\} = \{3,6,7,9\}$ and $a_1+1 = b_1$. Now (19) becomes $7 \equiv 2\mathtt{R}+\mathtt{T} \pmod{10}$ and there is only one solution, $(\mathtt{R},\mathtt{T}) = (7,3)$. Thus (20) becomes $\mathtt{B} \equiv \mathtt{O}+6 \pmod{10}$ which cannot be fulfilled with the remaining digits 6 and 9. Therefore, $b_4 = 2$, $\mathtt{E} = 2$ and $\mathtt{U} = 9$.

Looking at (21) as an equation $\mathtt{S}+b_3 = 9+a_3$ we find five solutions for $(\mathtt{S};a_3,b_3)$: it is one of $(8;0,1)$, $(8;1,2)$, $(7;0,2)$, $(7;1,3)$, $(6;0,3)$.

With these values for $\mathtt{S}$ the range of b_1 can be reduced to $1 \le b_1 \le 3$.

The next three steps require a little bit of straightforward computation. First, the three values of $\mathtt{S}$ create via (18) eleven possible solutions for $(\mathtt{X},\mathtt{G})$: $(0,4)$, $(3,0)$, $(5,4)$, $(7,8)$, $(8,0)$ if $\mathtt{S} = 6$; $(0,6)$, $(5,6)$, $(6,8)$ if $\mathtt{S} = 7$; and $(3,4)$, $(4,6)$, $(6,0)$ if $\mathtt{S} = 8$.

Second, congruence (19) contains the letters $\mathtt{R}$ and $\mathtt{T}$ and is transformed into the form $5-(b_1-a_1) \equiv 2\mathtt{R}+\mathtt{T} \pmod{10}$. As $a_1 \in \{0,1\}$ and $1 \le b_1 \le 3$ their difference is in $\{0,1,2,3\}$. Now we write down the solutions to the four possible congruences 2, 3, 4, or $5 \equiv 2\mathtt{R}+\mathtt{T}$ as

$(\mathtt{R}, \mathtt{T})$ pairs without using the values of $\mathtt{M}$, $\mathtt{E}$, and $\mathtt{U}$ and with $\mathtt{T} > 0$. For example, $5 \equiv 2\mathtt{R} + \mathtt{T}$ generates three pairs for $(\mathtt{R}, \mathtt{T})$: $(0, 5)$, $(4, 7)$, and $(6, 3)$. Next, two passes over each $(\mathtt{S}, \mathtt{X}, \mathtt{G})$ triple are made to eliminate first all $(\mathtt{R}, \mathtt{T})$ pairs from the generic list that conflict with the triple. Second the real values of a_1 and b_1 are computed and checked against the assumed difference $b_1 - a_1$. For example, $(\mathtt{S}, \mathtt{X}, \mathtt{G}) = (6, 3, 0)$ has only one candidate for $b_1 - a_1 = 0$: $(\mathtt{R}, \mathtt{T}) = (4, 7)$. As $\mathtt{G} + \mathtt{R} = 4$ we find $a_1 = 0$ and $2\mathtt{X} + \mathtt{R} + 2\mathtt{S} + \mathtt{E} = 24$ gives $b_1 = 2$ so that $b_1 - a_1 = 2$ which does not equal the assumed difference. Thus the candidate cannot give a valid solution. In total 18 candidates are found that might be valid assignments to the eight letters $\mathtt{T}$, $\mathtt{U}$, $\mathtt{G}$, $\mathtt{M}$, $\mathtt{E}$, $\mathtt{R}$, $\mathtt{X}$, and $\mathtt{S}$.

Third, we compute the *sum* $\mathtt{T} + 2\mathtt{E} + \mathtt{S} + b_2 - a_2$ for the 18 candidates and try to assign the remaining digits to $\mathtt{B}$ and $\mathtt{O}$ such that by (20) $\mathtt{B} \equiv \mathtt{O} + sum$, i.e., the difference between the larger and the smaller digit must equal either *sum* or $10 - sum$.

Here are the 18 tuples with the associated set $\{\mathtt{B}, \mathtt{O}\}$, the *sum*, and the pair $(\mathtt{B}, \mathtt{O})$ if there is a match: $(\mathtt{T}, \mathtt{U}, \mathtt{G}, \mathtt{M}, \mathtt{E}, \mathtt{R}, \mathtt{X}, \mathtt{S}; \{\mathtt{B}, \mathtt{O}\}; sum; (\mathtt{B}, \mathtt{O}))$ is $(8, 9, 4, 1, 2, 3, 0, 6; \{5, 7\}; 9; -)$, $(5, 9, 0, 1, 2, 4, 3, 6; \{7, 8\}; 6; -)$, $(7, 9, 0, 1, 2, 8, 3, 6; \{4, 5\}; 9; (4, 5))$, $(3, 9, 4, 1, 2, 0, 5, 6; \{7, 8\}; 3; -)$, $(7, 9, 4, 1, 2, 3, 5, 6; \{0, 8\}; 8; (8, 0))$, $(7, 9, 4, 1, 2, 8, 5, 6; \{0, 3\}; 9; -)$, $(3, 9, 8, 1, 2, 0, 7, 6; \{4, 5\}; 3; -)$, $(3, 9, 8, 1, 2, 5, 7, 6; \{0, 4\}; 4; (4, 0))$, $(5, 9, 8, 1, 2, 4, 7, 6; \{0, 3\}; 6; -)$, $(4, 9, 6, 1, 2, 5, 0, 7; \{3, 8\}; 6; -)$, $(8, 9, 6, 1, 2, 3, 5, 6; \{4, 5\}; 0; -)$, $(3, 9, 6, 1, 2, 0, 5, 7; \{4, 8\}; 4; -); (8, 4))$, $(3, 9, 8, 1, 2, 0, 6, 7; \{4, 5\}; 4; -)$, $(3, 9, 8, 1, 2, 5, 6, 7; \{0, 4\}; 5; -)$, $(5, 9, 8, 1, 2, 4, 6, 7; \{0, 3\}; 7; (0, 3))$, $(3, 9, 6, 1, 2, 0, 4, 8; \{5,7\}; 5; -)$, $(3, 9, 6, 1, 2, 5, 4, 8; \{0,7\}; 6; -)$, $(7, 9, 6, 1, 2, 3, 4, 8; \{0,5\}; 0; -)$.

Only five tuples remain for $(\mathtt{T}, \mathtt{U}, \mathtt{G}, \mathtt{M}, \mathtt{E}, \mathtt{B}, \mathtt{R}, \mathtt{X}, \mathtt{S}, \mathtt{O})$: $(3, 9, 8, 1, 2, 4, 5, 7, 6, 0)$ with $(a_3, b_3) = (0, 1)$; $(a_3, b_3) = (1, 2)$ for $(7, 9, 4, 1, 2, 8, 3, 5, 6, 0)$ and $(3, 9, 6, 1, 2, 8, 0, 5, 7, 4)$; $(a_3, b_3) = (1, 3)$ is computed for $(7, 9, 0, 1, 2, 4, 8, 3, 6, 5)$; and $(5, 9, 8, 1, 2, 0, 4, 6, 7, 3)$ has $(a_3, b_3) = (0, 2)$. The comparison with the five solutions for $(\mathtt{S}; a_3, b_3)$ above shows that only the last one has an acceptable combination $(\mathtt{S}; a_3, b_3) = (7; 0, 2)$. Only this one can be a solution of the alphametic.

The alphametic has a unique solution: $(\mathtt{T}, \mathtt{U}, \mathtt{G}, \mathtt{M}, \mathtt{E}, \mathtt{B}, \mathtt{R}, \mathtt{X}, \mathtt{S}, \mathtt{O}) = (5, 9, 8, 1, 2, 0, 4, 6, 7, 3)$.