"PERFECT" MAGIC SQUARES

Jeremiah and Karen Farrell

Martin Gardner's informative *Scientific American* essay "Perfect, Amicable, Sociable" is updated in his 1977 book Mathematical Magic Show (see Appendix). In the essay Gardner asserts:

> ONE WOULD BE hard put to find a set of whole numbers with a more fascinating history and more elegant properties, surrounded by greater depths of mystery—and more totally useless—than the perfect numbers and their close relatives, the amicable (or friendly) numbers.
>
> A perfect number is simply a number that equals the sum of its proper divisors; that is, of all its divisors except itself. The smallest such number is 6, which equals the sum of its three divisors, 1, 2, and 3. The next is 28, the sum of $1 + 2 + 4 + 7 + 14$. Early commentators on the Old Testament, both Jewish and Christian, were much impressed by the perfection of those two numbers. Was not the world created in six days and does not the moon circle the earth in twenty-eight? In *The City of God*, Book 11, Chapter 30, St. Augustine argues that although God could easily have created the world in an instant, He preferred to take six days because the perfection of 6 signifies the perfection of the universe. (Similar views had been advanced earlier by the first-century Jewish philosopher Philo Judaeus in the third chapter of his *Creation of the World*.)

Euclid had proven that $2^{n-1}(2^n-1)$ was always a perfect number whenever (2^n-1) was prime and 1000 years later Leonard Euler proved that all even perfects were given by this formula. No odd perfect numbers are believed to exist and only finitely many evens are now known.

Primes of the form 2^n-1 are called Mersenne primes after a seventeenth century mathematician who studied them.

Amicable numbers are generalizations of perfect numbers. These are pairs in which the sum of the divisors of one is equal to the other. The smallest pair are 220 and 284. This pair was known to the Pythagorians.

If the chain that leads back to the original number has more than two links the number is called "sociable".

By 1977 only 24 perfect numbers were know and Gardner's list follows.

	FORMULA	NUMBER	NUMBER OF DIGITS
1	$2^{1}(2^{2}-1)$	6	1
2	$2^{2}(2^{3}-1)$	28	2
3	$2^{4}(2^{5}-1)$	496	3
4	$2^{6}(2^{7}-1)$	8,128	4
5	$2^{12}(2^{13}-1)$	33,550,336	8
6	$2^{16}(2^{17}-1)$	8,589,869,056	10
7	$2^{18}(2^{19}-1)$	137,438,691,328	12
8	$2^{30}(2^{31}-1)$	2,305,843,008,139,952,128	19
9	$2^{60}(2^{61}-1)$		37
10	$2^{88}(2^{89}-1)$		54
11	$2^{106}(2^{107}-1)$		65
12	$2^{126}(2^{127}-1)$		77
13	$2^{520}(2^{521}-1)$		314
14	$2^{606}(2^{607}-1)$		366
15	$2^{1,278}(2^{1,279}-1)$		770
16	$2^{2,202}(2^{2,203}-1)$		1,327
17	$2^{2,280}(2^{2,281}-1)$		1,373
18	$2^{3,216}(2^{3,217}-1)$		1,937
19	$2^{4,252}(2^{4,253}-1)$		2,561
20	$2^{4,422}(2^{4,423}-1)$		2,663
21	$2^{9,688}(2^{9,689}-1)$		5,834
22	$2^{9,940}(2^{9,941}-1)$		5,985
23	$2^{11,212}(2^{11,213}-1)$		6,751
24	$2^{19,936}(2^{19,937}-1)$		12,003

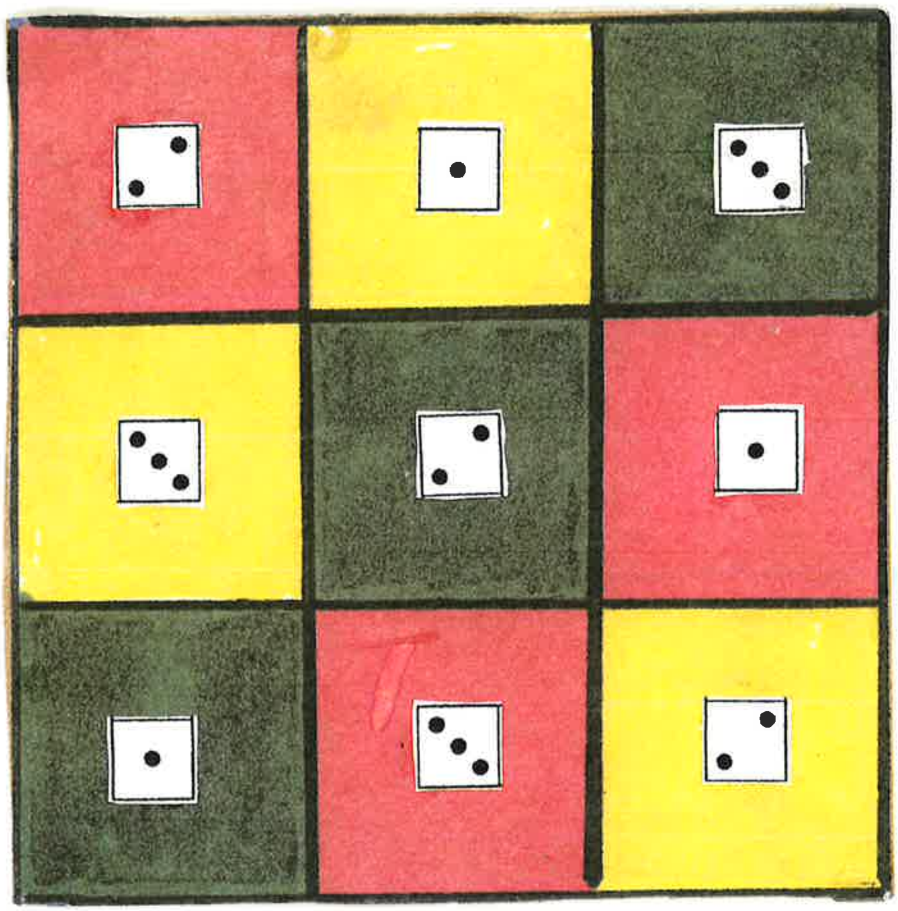
The twenty-four known perfect numbers

In April 1979 David Slowinski and Harry Nelson found the 27th Mersenne prime of 13,395 digits. Nelson signed for me a copy of this prime at a Gathering for Gardner meeting in Atlanta.

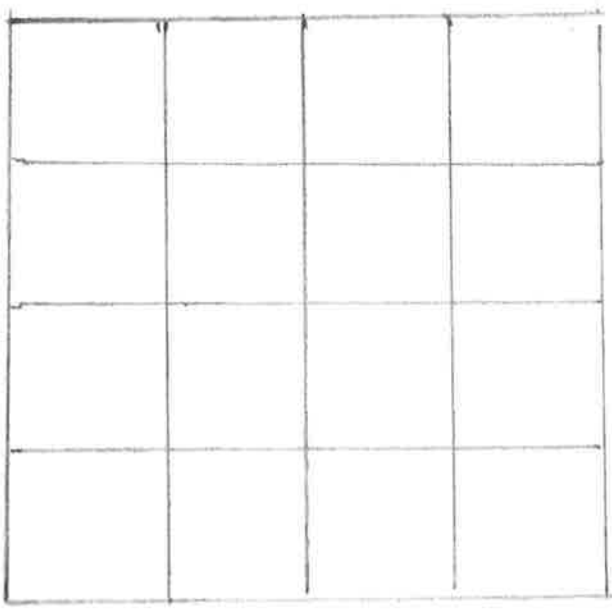

THE 27TH MERSENNE PRIME

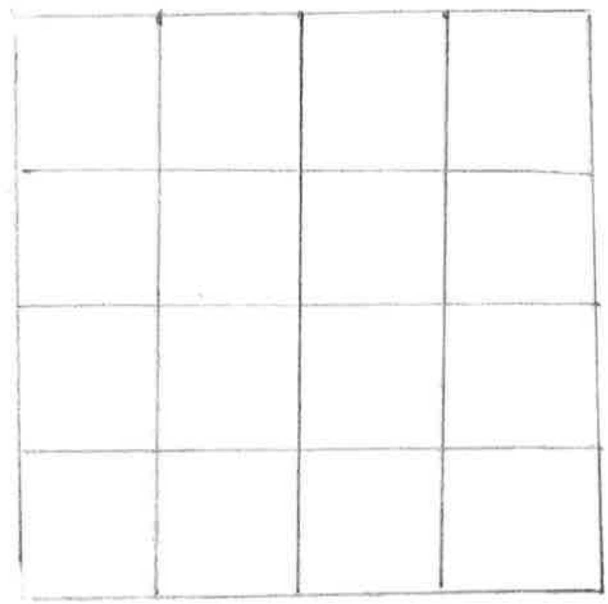
As of "D-Day" 2020 the number of known perfects was 51 which has 49,724,095 digits with a Mersenne prime of 24,862,048 digits.

A Perfect Magic Square will have as constant a perfect number. For example here is a 3x3 with constant 6.

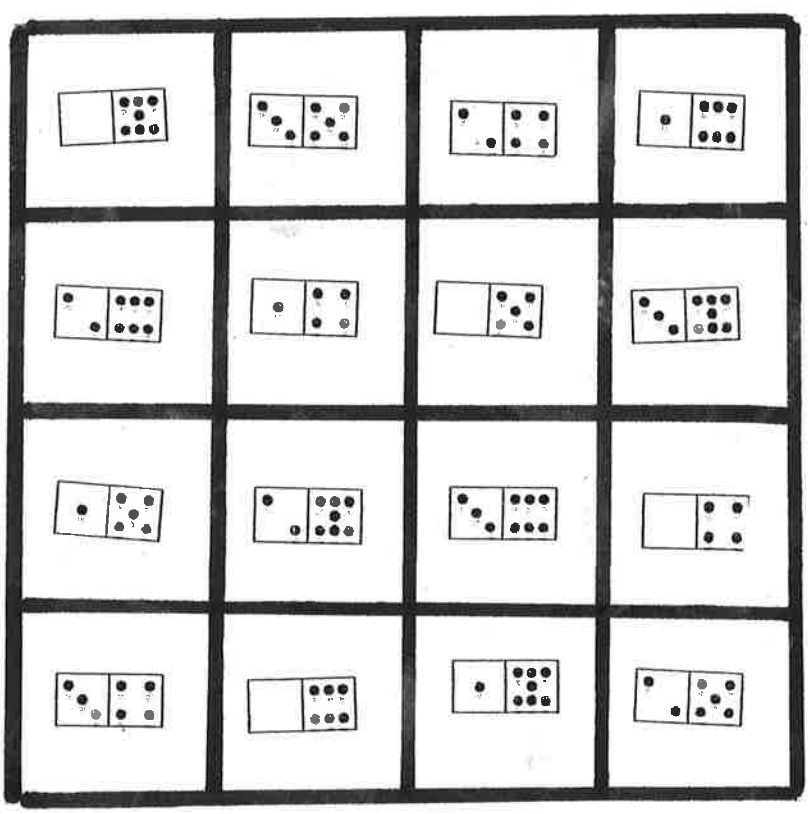


If we allow the numbers 0, 1, 2 and 3 to be used in a 4x4 we ask the reader to use each number four times to fill the square with constant sum 6 in every row, column, and all eight diagonals including the broken 2x2s and 3x1s. As another puzzle use the letters NODE in a 4x4 so that legitimate words can be anagrammed in all sixteen sets of four.





An example of a perfect 28 using 16 different dominoes from a double-nine set follows.

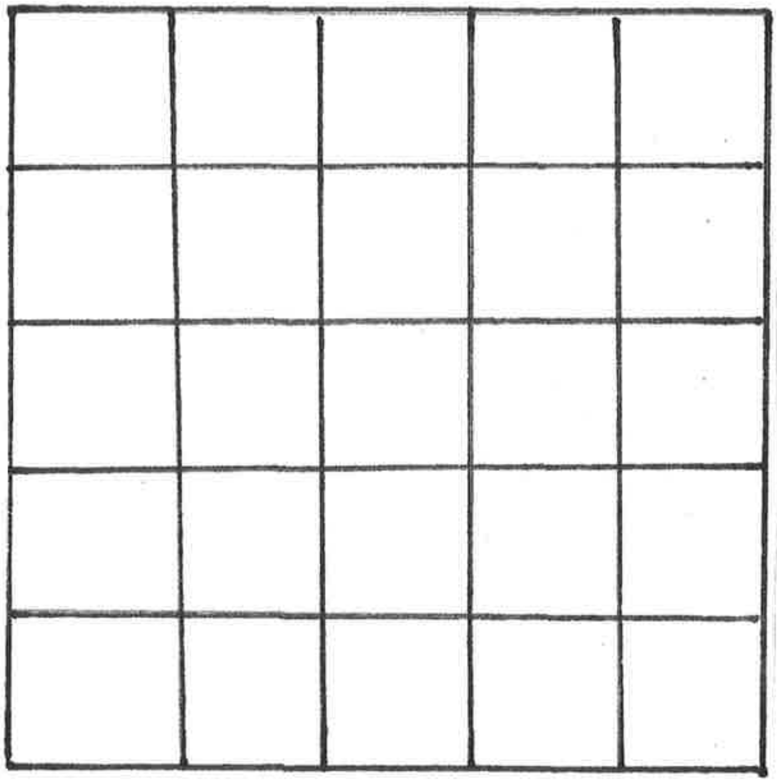


Gardner's popular work "Annotated Alice" has now grown to be more wordy than the original Lewis Carroll Alices. As an exercise place five copies of ALICE with five copies of 1, 2, 4, 7, and 14 so that all rows, columns and diagonals anagram to ALICE and sum to 28. Also we insist that in the five copies of each letter of ALICE all five numbers are used.

In the next example place four each of the numbers 120, 123, 125 and 128 into the 4x4 square to make it magic with constant 496, the third perfect.

Another example. Place 25 different dominoes from a double-nine set so that the result is magic with constant 45. Then add 90.2 to each entry to obtain a magic sum of 496.

Another problem. Use each letter of the word IMPERSONAL five times in a 5x5 to form meaningful pairs as entries that magically anagram into the word IMPERSONAL.



Our answer to IMPERSONAL uses pairs from Chamber's Dictionary with 16 chemical elements and 9 others with only SL meaning "Solitictor at Law" unusual.

Other magic squares with interesting constants are of course possible.

One example we like uses 2, 3, 8, 15 with constant 28.

2	**8**	**15**	**3**
15	**3**	**2**	**8**
3	**15**	**8**	**2**
8	**2**	**3**	**15**

Another that might interest young children could use the 16 playing cards aces, twos, threes and fours. They would be asked to arrange the cards into a 4x4 square with constant 10. One of many solutions follows with a red-black checkerboard arrangement.

AS	**4D**	**3S**	**2D**
3H	**2C**	**AH**	**4C**
2S	**3D**	**4S**	**AD**
4H	**AC**	**2H**	**3C**

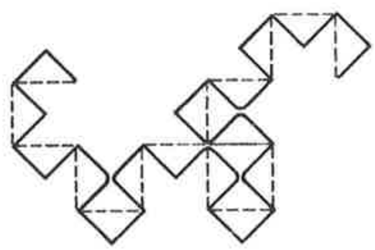
REFERENCE PAGE

MARTIN GARDNER

Martin Gardner

Mathematical Magic Show

MORE PUZZLES, GAMES, DIVERSIONS, ILLUSIONS & OTHER MATHEMATICAL SLEIGHT-OF-MIND FROM *SCIENTIFIC AMERICAN*



WITH REPARTEE FROM READERS, AFTERTHOUGHTS FROM THE AUTHOR AND 133 DRAWINGS & DIAGRAMS

New York 1977
ALFRED A. KNOPF

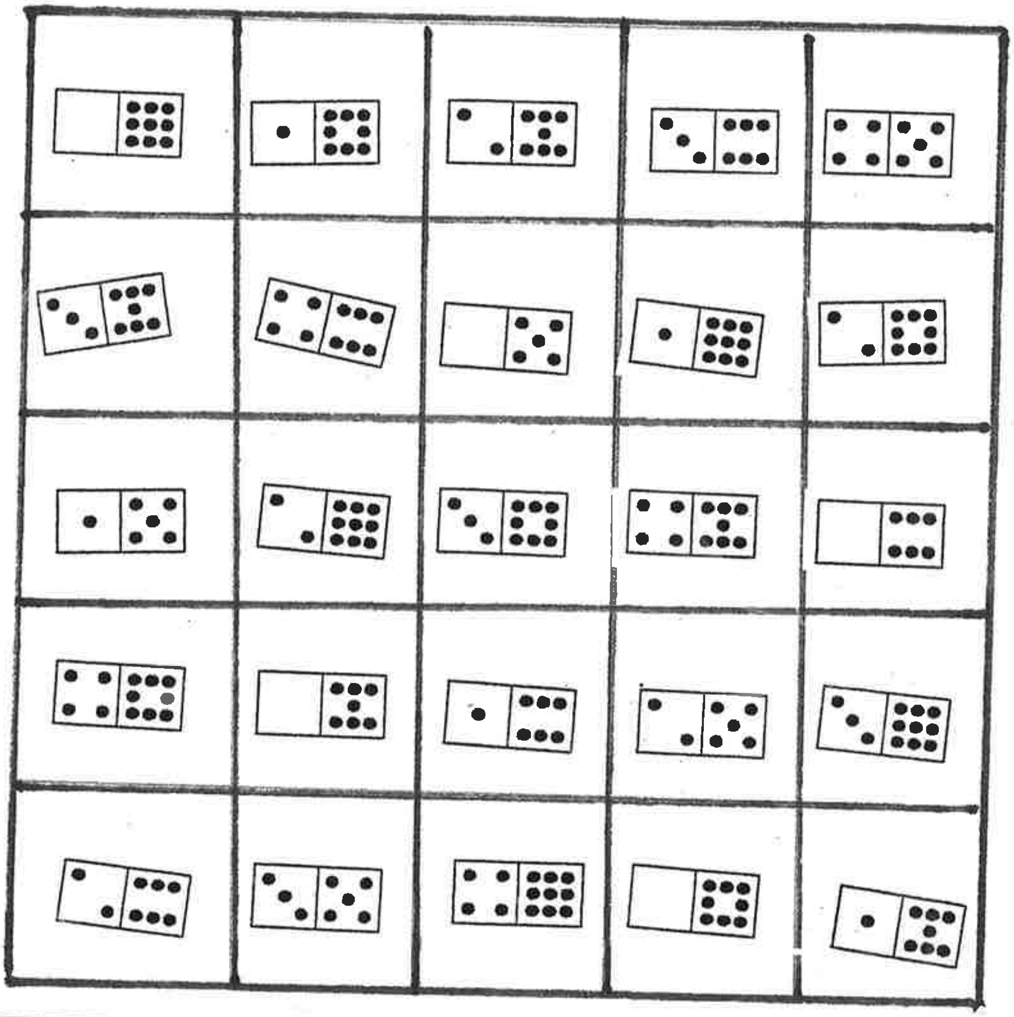
SOME POSSIBLE ANSWERS

2	1	0	3
0	3	2	1
3	0	1	2
1	2	3	0

N	O	D	E
D	E	N	O
E	D	O	N
O	N	E	D

A 1	L 7	I 2	C 14	E 4
C 2	E 14	A 4	L 1	I 7
L 4	I 1	C 7	E 2	A 14
E 7	A 2	L 14	I 4	C 1
I 14	C 4	E 1	A 7	L 2

125	123	120	128
120	128	125	123
128	120	123	125
123	125	128	120



PA	ME	IN	LO	SR
NO	SL	AR	PE	MI
ER	PI	MO	SN	AL
SM	NA	EL	IR	PO
LI	OR	PS	AM	NE