2006

Spanning Eulerian Subgraphs in claw-free graphs

Zhi-Hong Chen
Butler University, chen@butler.edu

Hong-Jian Lai

Weiqi Luo

Yehomg Shao

Spanning Eulerian Subgraphs in claw-free graphs

Zhi-Hong Chen
Butler University, Indianapolis, IN 46208

Hong-Jian Lai
West Virginia University, Morgantown, WV 26506

Weiqi Luo
JiNan University, Guangzhou, P.R. China

Yehong Shao
Arts and Sciences, Ohio University Southern, Ironton, OH 45638

Abstract

A graph is claw-free if it has no induced $K_{1,3}$ subgraph. A graph is essential 4-edge-connected if removing at most three edges, the resulting graph has at most one component having edges. In this note, we show that every essential 4-edge-connected claw free graph has a spanning Eulerian subgraph with maximum degree at most 4.

The graphs in this paper are finite and loopless. For terms not defined here, see Bondy and Murty [1]. Let G be a graph. Then $\delta(G)$ denotes the minimum degree of G. An edge subset $X \subseteq E(G)$ is an *essential edge-cut* of G if each component of $G - X$ has at least an edge. A graph G is *essentially k-edge-connected* if for any $X \subseteq E(G)$ with $|X| < k$, at most one component of $G - X$ has edges. The line graph $L(G)$ of a graph G has $E(G)$ as its vertex set and two vertices of $L(G)$ are adjacent if and only if they are adjacent as edges in G. A graph is called *claw-free* if it has not induced subgraph isomorphic to $K_{1,3}$. It is well-known that a line graph is claw-free; and that for a graph G that is not isomorphic to a $K_{1,n-1}$, G is essentially k-edge-connected if and only if $L(G)$ is k-connected. If G has a cycle containing every vertex of G, then G is called *Hamiltonian.* A graph G is *even* if every vertex of G has even degree. A graph is *Eulerian* if it is connected even graph. A graph with a spanning Eulerian subgraph H with $\Delta(H) = 2$ is Hamiltonian. A graph is *supereulerian* if it has a spanning Eulerian subgraph. An Eulerian subgraph H of G is *dominating* if $E(G - V(H)) = \emptyset$. Harary and Nash-Williams [3] showed that for a connected graph G with $|E(G)| \geq 3$, $L(G)$ is Hamiltonian if and only if G contains a dominating Eulerian subgraph.

The following conjectures are well known.

Conjecture (Matthews and Sumner [4]). Every 4-connected claw-free graph is Hamiltonian.

Conjecture (Thomassen [6]). Every 4-connected line graphs is Hamiltonian.

Ryjáček [5] proved that these two conjectures are equivalent. In this paper, we present a best possible result on claw-free graph with spanning Eulerian subgraphs with maximum degree at most 4.

In [2], Catlin introduced the concept of collapsible graphs. A graph G is called *collapsible* if for every even subset $S \subseteq V(G)$, there is a subgraph H such that $G - E(H)$ is connected and the set of all odd degree vertices in H is equal to S. Note that if a graph G is collapsible, then G is supereulerian. By definition, K_1 is both collapsible and supereulerian graph.

Catlin showed that every graph G has a unique collection of pairwise vertex-disjoint maximal collapsible subgraphs $H_1, \cdots, H_c$ such that $\bigcup_{i=1}^{c} V(H_i) = V(G)$. Let G be a graph and let $X \subseteq E(G)$. The contraction G/X is the graph obtained from G by identifying the ends of each edge in X and deleting the resulting loops. If H is a subgraph of G, then we use G/H for $G/E(H)$. Combining the concepts of collapsible graphs and the contraction, Catlin [2] define the reduction of a graph. The *reduction* of G, denoted by G', is the graph obtained from G by contracting each maximal collapsible subgraph H_i, $(1 \leq i \leq c)$, into a single vertex v_i. A graph G is *reduced* if $G = G'$. Each vertex v in G' defined the maximal collapsible subgraph H in G to be the preimage of v in the contraction. A vertex in G' is a trivial contraction if its preimage is K_1.

Theorem A (Catlin [2]). Let G be a graph and let G' be the reduction of G. Then each of the following holds.
(1) G is collapsible if and only if $G' = K_1$.
(2) G is supereulerian if and only if G' is supereulerian.
(3) If G is reduced, then G is K_3-free with $\delta(G) \leq 3$.

Theorem 1. Let G be an essential 4-edge-connected claw-free simple graph with $\delta(G) \geq 3$. Then
(1) G is collapsible, and
(2) G has spanning Eulerian subgraph H with $\Delta(H) \leq 4$.

Proof. We prove (1) first. By way of contradiction, suppose that G is not collapsible. Let G' be the reduction of G. By Theorem A (3), $\delta(G') \leq 3$. Let v be a vertex in G' with $\delta(G') = d_{G'}(v)$, and let H be the preimage of v in G. Since $\delta(G) \geq 3$, and since G is essentially 4-edge-connected, we must have $d_{G'}(v) \geq 3$, where equality holds only if $|V(H)| = 1$.

If $d_{G'}(v) = 3$, then since G is claw-free, two of the three edges incident with v will be in a triangle. It follows that $|V(H)| \geq 3$, contrary to the fact that $|V(H)| = 1$. Therefore, $\delta(G') \geq 4$, contrary to Theorem A(3), and so G must be collapsible.

Next, we prove (2) of the theorem.

Since G is collapsible, G has spanning Eulerian subgraphs. Let H be a spanning Eulerian subgraph of G with the edge set $|E(H)|$ as small as possible. We shall show that $\Delta(H) \leq 4$.

By way of contradiction, there is a vertex v in $V(H)$ such that $d_H(v) = \Delta(H) \geq 6$. Then there are at least 6 vertices adjacent to v in H. Note that since H is Eulerian simple graph, each component of $H - v$ has nonzero even number of vertices adjacent to v in H.

Claim 1. Suppose that $A = \{v_1, v_2, \cdots v_r\}$ is the set of vertices in a component of $H - v$ such that each v_i is adjacent to v in H, and that $r \geq 4$. For any pair of distinct v_i, $v_j \in A$, if $v_iv_j \in E(G)$ then $v_iv_j \in E(H)$.

If there are $i \neq j$ such that $v_iv_j \in E(G) - E(H)$, then since v_i and v_j with other vertices are in the same component of $H - v$, $H_0 = H + v_iv_j - \{vv_i, vv_j\}$ is connected and spanning even subgraph of G. Thus, H_0 is a spanning Eulerian subgraph of G with fewer edges than H, a contradiction. Claim 1 holds.

Case 1. One component of $H - v$ has at least six vertices adjacent to v in H.

Suppose that v_1, v_2, v_3, v_4, v_5 and v_6 are six vertices in a component of $H - v$ such that each v_i is adjacent to v in H. Let $S = \{v, v_1, v_2, v_3, v_4, v_5, v_6\}$. Let $H[S]$ be the induced subgraph of H by S, and let G_a be the graph obtained from the 5 vertex wheel W_4 by removing a rim edge from W_4 (see Figure 1).

Claim 2. $H[S]$ contains a subgraph isomorphic to K_4 or G_a with the given vertices where $\{a, b, c, d\} \subseteq S$ (Figure 1).

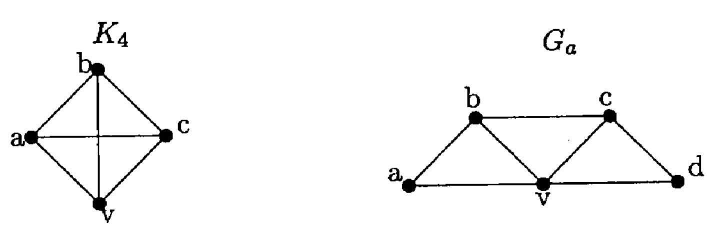

Figure 1

Since G is claw-free, any three edges incident with v will form a claw, and so at least two of these three edges must be in a K_3. By Claim 1, any triangle K_3 with two edges incident with v is a triangle in H. Claim 2 follows after repeated applications of this fact to the subgraph induced by vertex v and any three vertices in $S-\{v\}$.

By Claim 2, H contains either K_4 or G_a as shown in Figure 1. Let $H[v,b,c]$ be the triangle formed by the three vertices v, b, c in either K_4 or G_a. Then $H_v = H - \{vb, vc, bc\}$ is a spanning Eulerian subgraph of G with fewer edges than H, a contradiction.

Case 2. One component of $H - v$ has four vertices adjacent to v in H.

Let H_1 be a component of $H - v$ which has four vertices, say v_1, v_2, v_3 and v_4, adjacent to v in H. By Claim 1, any $v_iv_j \in E(G)$ implies that $v_iv_j \in E(H)$. Since $d_H(v) \geq 6$, v is a cut vertex, and so $H - v$ has another component, say H_2, which should has at least two vertices, say u_1 and u_2, adjacent to v in H.

Subcase 1. There are two vertices, say v_1 and v_2, in $\{v_1, v_2, v_3, v_4\}$ such that $v_1v_2 \notin E(G)$.

Let $G[v, v_1, v_2, u_2]$ be the induced subgraph by $\{v, v_1, v_2, u_1\}$. Since G is claw free, and $v_1v_2 \notin E(G)$, either u_1v_1 or u_1v_2 must be an edge in $G - E(H)$. We may assume that $u_1v_1 \in E(G) - E(H)$. Then $H_0 = H + u_1v_1 - \{vu_1, vv_1\}$ is connected and spanning even subgraph of G. Thus H_0 is a spanning Eulerian subgraph of G with fewer edges than H, a contradiction.

Subcase 2. Any two vertices in $\{v_1, v_2, v_3, v_4\}$ are adjacent to each other in H.

Then the induced subgraph $G[v, v_1, v_2, v_3, v_4] \cong K_5$. Then $H_0 = H - \{vv_1, vv_2, v_1v_2\}$ is spanning Eulerian subgraph of G with fewer edges than H, a contradiction again.

Case 3. Each component has only two vertices adjacent to v in H.

Since $d_H(v) \geq 6$, $H - v$ has at least three components, say H_1, H_2 and H_3. Suppose that $x_i \in V(H_i)$ is a vertex adjacent to v in H

$(i = 1, 2, 3)$. These three edges incident with v forms a claw. Since G is claw free, we may assume that $x_1x_2 \in E(G) - E(H)$. Then $H_0 = H + x_1x_2 - \{vx_1, vx_2\}$ is spanning Eulerian subgraph of G with fewer edges than H, a contradiction.

Since each case leads to a contradiction, this shows that $d_H(v) \leq 4$, and so $\Delta(H) \leq 4$. Theorem 1 is proved. □.

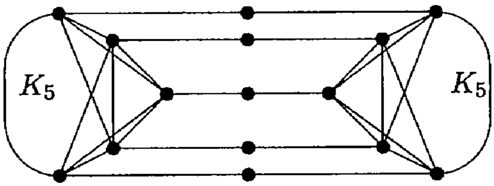

Figure 2

To see that Theorem 1 is best possible, let G be a graph obtained by connecting two complete graphs K_r $(r \geq 5)$ with 5 paths of length two in such a way that each end of a path is incident with one vertex in K_r (see Figure 2 for the case $r = 5$). Therefore, G is an essentially 4-edge-connected claw free graph with $\delta(G) = 2$. However, the reduction of G is $K_{2,5}$ which has no spanning Eulerian subgraphs and so G is not collapsible. This shows that the condition $\delta(G) \geq 3$ in Theorem 1 is necessary. Also the result is best possible in the following sense. Let G be a graph obtained from a path of length $r \geq 3$ by replacing each edge in the path by a complete graph of order 4. Then G is an essentially 4-edge-connected claw free graph with $\delta(G) \geq 3$. Obviously, G cannot have a spanning Eulerian subgraph with $\Delta(G) = 2$. The best we can have is a spanning Eulerian subgraph with $\Delta(G) = 4$.

The proof of Theorem 1 also yields the following.

Proposition 2. Let G be a claw-free graph. If G has a spanning Eulerian subgraph, then G has a spanning Eulerian subgraph H with $\delta(H) \leq 4$.

References

[1] J. A. Bondy and U. S. R. Murty, "Graph Theory with Applications". American Elsevier, New York (1976).

[2] P. A. Catlin, A reduction method to find spanning eulerian subgraphs. J. Graph Theory 12 (1988) 29-44.

[3] F. Harary and C. S. J. A. Nash-Williams, On eulerian and hamiltonian graphs and line graphs. Canad. Math. Bull. 8 (1965) 701-710.

[4] M. M. Matthews and D. P. Sumner, Hamiltonian results in $K_{1,3}$-free graphs, J. Graph Theory, 8 (1984) 139-146.

[5] Z. Ryjáček, On a closure concept in claw-free graphs, J. Combin. Theory Ser. B. , 70 (1997) 217-224.

[6] C. Thomassen, Reflections on graph theory. J. Graph Theory 10 (1986) 309-324.