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Spanning trails containing given edges

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Abstract

A graph $G$ is Eulerian-connected if for any $u$ and $v$ in $V(G)$, $G$ has a spanning $(u, v)$-trail. A graph $G$ is edge-Eulerian-connected if for any $e'$ and $e''$ in $E(G)$, $G$ has a spanning $(e', e'')$-trail. For an integer $r \geq 0$, a graph is called $r$-Eulerian-connected if for any $X \subseteq E(G)$ with $|X| \leq r$, and for any $u, v \in V(G)$, $G$ has a spanning $(u, v)$-trail $T$ such that $X \subseteq E(T)$. The $r$-edge-Eulerian-connectivity of a graph can be defined similarly. Let $\theta(r)$ be the minimum value of $k$ such that every $k$-edge-connected graph is $r$-Eulerian-connected. Catlin proved that $\theta(0) = 4$. We shall show that $\theta(r) = 4$ for $0 \leq r \leq 2$, and $\theta(r) = r + 1$ for $r \geq 3$. Results on $r$-edge-Eulerian connectivity are also discussed.

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1. Introduction

We follow the notation of Bondy and Murty [1], except that graphs have no loops. A graph $G$ is Hamiltonian-connected if for every pair of vertices $u, v$ of $G$, there is a Hamiltonian $(u, v)$-path in $G$. For a graph $G$, a trail is a vertex-edge alternating sequence $v_0, e_1, v_1, e_2, \ldots, e_{k-1}, v_{k-1}, e_k, v_k$ such that all the $e_i$'s are distinct and $e_i = v_{i-1}v_i$ for all $i$. Let $e', e'' \in E(G)$. A trail in $G$ whose first edge is $e'$ and whose last edge is $e''$ is called an $(e', e'')$-trail. For $u, v \in V(G)$, a $(u, v)$-trail of $G$ is a trail in $G$ whose origin is $u$ and whose terminus is $v$. A trail $H$ is called a dominating trail of $G$ if every edge of $G$ is incident with at least one vertex of $H$ in $G$. A trail $H$ is called a spanning trail if $V(H) = V(G)$. If $u = v$, then a $(u, v)$-trail in $G$ is a closed trail, which is also called a Eulerian subgraph of $G$. A graph is called supereulerian if it has a spanning closed trail. The collection of all supereulerian graphs is denoted by $\Sigma$. A graph $G$ is Eulerian-connected if for any $u, v$ in $V(G)$ (including the case $u = v$), $G$ has a spanning $(u, v)$-trail. A graph is called $r$-Eulerian-connected if for any $X \subseteq E(G)$ with $|X| \leq r$, and for any $u, v \in V(G)$, $G$ has a spanning...
(u, v)-trail T such that X ⊆ E(T). For an integer r ≥ 0, the collection of all r-Eulerian-connected graphs is denoted by \( \mathcal{L}(r) \). Obviously, \( \mathcal{L}(r) \subseteq \mathcal{F}(r) \) for all r ≥ 0.

A graph G is edge-Eulerian-connected if for any e’, e'' in E(G), G has a spanning (e’, e'')-trail. A graph is called r-edge-Eulerian-connected if for any X ⊆ E(G) with |X| ≤ r and for any e’, e'' ∈ E(G), G has a spanning (e’, e'')-trail T such that X ⊆ E(T). For an integer r ≥ 0, the collection of all r-edge-Eulerian-connected graphs is denoted by \( \mathcal{L}(r) \).

Many studies have been done on Eulerian graphs (see [7]). For the literature on the subject of supereulerian graphs, see surveys [3,6]. Harary and Nash-Williams [9] demonstrated the relationship between Eulerian subgraphs and Hamiltonian cycles in the line graph of G. Zhan [14] studied (e’, e'')-trails of a graph G for the Hamiltonian connectivity of the line graph of G. In the study of spanning trails of graphs [2], Catlin introduced the concept of collapsible graphs.

For a graph G, let O(G) be the set of odd degree vertices of G and let R be an even subset of V(G). A subgraph H_R of G is called a spanning R-trail if H_R is a spanning connected subgraph such that O(H_R) = R. A graph G is collapsible if for every even subset R ⊆ V(G), G has a spanning R-trail. We will regard an empty set as an even subset and K_1 as both collapsible and supereulerian. The collection of all collapsible graphs is denoted by \( \mathcal{E}(r) \). By the definition of collapsible graphs, we have:

**Proposition A.** Let G be a collapsible graph. Then each of the following holds

(i) G is supereulerian.

(ii) G is Eulerian-connected.

**Proof.** For any vertices u, v ∈ V(G). Let R = ∅ if u = v, or R = {u, v} if u ≠ v. Since G is collapsible, it has a spanning subgraph H_R such that O(H_R) = R. Therefore, H_R is a spanning Eulerian subgraph of G if R = ∅, or H_R is a (u, v)-spanning trail of G. □

Let X ⊆ E(G) and let R be an even subset of V(G). A spanning R-trail H_R of G such that X ⊆ E(H_R) is called a spanning (R, X)-trail, and denoted by H_R(X). A graph is called strongly r-Eulerian-connected if for any X ⊆ E(G) with |X| ≤ r and for any even subset R ⊆ V(G), G has a spanning R-trail H_R such that X ⊆ E(H_R) (i.e. G has a H_R(X)). The collection of all strongly r-Eulerian-connected graphs is denoted by \( \mathcal{E}(r) \).

For an integer r, define \( \mathcal{L}(r) \) to be the family of graphs such that G ∈ \( \mathcal{L}(r) \) if and only if for any subset X ⊆ E(G) with |X| ≤ r, G has a spanning Eulerian subgraph H such that X ⊆ E(H). Define f(r) to be the minimum value of k such that every k-edge-connected graph G is in \( \mathcal{L}(r) \). In [12], Lai found f(r) for all the values of r (see Corollary 3.6). Let \( \theta(r) \) be the minimum value of k such that every k-edge-connected graph is in \( \mathcal{L}(r) \) and let \( \psi(r) \) be the minimum value of k such that every k-edge-connected graph is in \( \mathcal{E}(r) \). Since \( \mathcal{E}(r) \subseteq \mathcal{L}(r) \subseteq \mathcal{L}(r) \),

\[
f(r) \leq \theta(r) \leq \psi(r).
\]

Let \( \zeta(r) \) be the minimum value of k such that every k-edge-connected graph is in \( \mathcal{E}(r) \). In this paper, we will determine the values of \( \theta(r) \), \( \psi(r) \), and \( \zeta(r) \) for all r ≥ 0.

In the next section, we will present Catlin’s reduction method and some preliminary results which are needed in our proofs. Our main results are in Sections 3 and 4. We will present our results on r-Eulerian-connected graphs, and give the values of \( \theta(r) \) and \( \psi(r) \) for all r ≥ 0. Section 4 contains results on the r-edge-Eulerian connected graphs.

2. Catlin’s reduction method and preliminary results

Let H be a connected subgraph of G. The contraction G/H is obtained from G by contracting each edge of H and deleting the resulting loops. In [2], Catlin showed that every graph G has a unique collection of pairwise vertex-disjoint maximal collapsible subgraphs H_1, H_2, ..., H_k such that \( \bigcup_{i=1}^{k} V(H_i) = V(G) \). The reduction of G is obtained from G by contracting each of H_i into a vertex v_i for all i, and is denoted by G’. Each H_i is called a preimage of v_i in G, and v_i is called the contraction image of H_i in G’. A vertex v in G’ is called a trivial contraction if its preimage in G is K_1. A graph G is reduced if G is the reduction of some graph. Let F(G) be the minimum number of edges that must be added to G so that the resulting graph has 2 edge-disjoint spanning trees.
Theorem 2.1 (Catlin [2]). Let $G$ be a graph, and let $G'$ be the reduction of $G$. Each of the following holds.

(i) $G$ is supereulerian if and only if $G'$ is supereulerian.
(ii) $G$ is collapsible if and only if $G' \cong K_1$
(iii) $|E(G')| + F(G') = 2|V(G')| - 2$.

In [10], Jaeger proved that a graph with two edge-disjoint spanning trees is supereulerian. In [2], Catlin proved that if $G$ has two edge-disjoint spanning trees, then $G$ is collapsible. It is well known now that a $2k$-edge-connected graph has $k$ edge-disjoint spanning trees [8,11,13]. Thus, we have:

Theorem 2.2. If $G$ is 4-edge-connected, then $G$ is collapsible.

In [4], Catlin proved:

Theorem 2.3 (Catlin [4]). Let $G$ be a graph and let $k \geq 1$ be an integer. The following are equivalent:

(i) $G$ is $2k$-edge-connected;
(ii) For any $X \subseteq E(G)$ with $|X| \leq k$, $G - X$ has $k$ edge-disjoint spanning trees.

Corollary 2.4 (Catlin [4]). Let $G$ be a graph and let $k \geq 1$ be an integer. The following are equivalent:

(i) $G$ is $(2k + 1)$-edge-connected;
(ii) For any $X \subseteq E(G)$ with $|X| \leq k + 1$, $G - X$ has $k$-edge-disjoint spanning trees.

The following theorems will be needed in our proofs.

Theorem 2.5 (Catlin et al. [5]). Let $G$ be a connected graph. If $F(G) \leq 2$, then either $G$ is collapsible, or the reduction of $G$ is in $\{K_2, K_{2,t} : t \geq 1\}$.

Let $e$ be an edge in $G$. Edge $e$ is subdivided when it is replaced by a path of length 2 whose internal vertex, denoted by $v(e)$, has degree 2 in the resulting graph. The process of taking an edge $e$ and replacing it by that path of length 2 is called subdividing $e$. Let $G$ be a graph and let $X \subseteq E(G)$. Let $G_X$ be the graph obtained from $G$ by subdividing each edge in $X$. Then $V(G_X) = V(G) \cup \{v(e) \text{ for each } e \in X\}$.

Lemma 2.6. Let $k \geq 2$ be an integer. Let $G$ be a connected graph and let $X \subseteq E(G)$. Let $R$ be an even subset of $V(G)$. Then each of the following holds

(i) $G$ has a spanning $(R, X)$-trail $H_R(X)$ if and only if $G_X$ has a spanning $R$-trail. In particular, $G$ has a spanning closed trail $H$ such that $X \subseteq E(H)$ if and only if $G_X$ is supereulerian.
(ii) If $G_X$ is collapsible, then $G_X$ has a spanning $R$-trail.
(iii) Let $X = X_1 \cup X_2$ with $X_1 \cap X_2 = \emptyset$. Then $F(G_X) \leq F((G - X_1)_{X_2})$.
(iv) If $G$ has $k$ edge-disjoint spanning trees, then for any $X \subseteq E(G)$ with $|X| \leq 2k - 2$, $F(G_X) \leq 2$.

Proof. (i) and (ii) follow from the definitions of collapsibility and $G_X$.
(iii) Let $p = F((G - X_1)_{X_2})$. Let $E_p$ be the $p$ edge set such that $(G - X_1)_{X_2} + E_p$ has 2-edge-disjoint spanning trees $(T_1$ and $T_2)$. Let $X_1 = \{e_1, e_2, \ldots, e_s\}$ and each $e_i = u_i v_i$ ($1 \leq i \leq s$). By the definition of $G_X$, we know that $G_X$ can be obtained from $(G - X_1)_{X_2}$ by joining each pair of $u_i$ and $v_i$ by a path $P_i = u_i v(e_i) v_i$ where $v(e_i)$ is a new vertex. Therefore, $T_1 + \bigcup_{i=1}^{s} \{u_i v(e_i)\}$ and $T_2 + \bigcup_{i=1}^{s} \{v(e_i) v_i\}$ are two edge-disjoint spanning trees in $G_X + E_p$, and so $F(G_X) \leq p = F((G - X_1)_{X_2})$.
(iv) Let $T_1, T_2, \ldots, T_k$ be $k$ edge-disjoint spanning trees of $G$. Without lost of generality, we may assume that

\[ |X \cap E(T_1)| \leq |X \cap E(T_2)| \leq \cdots \leq |X \cap E(T_k)|. \]
Since \( k \geq 2, |X| \leq 2k - 2 \), \( T_i \)'s are edge-disjoint, and by (2),

\[
|X \cap E(T_1)| + |X \cap E(T_2)| \leq 2.
\] (3)

Let \( X = \{e_1, e_2, \ldots, e_p\} \) where \( p \leq 2k - 2 \), and let \( e_i = u_iv_i \) for all \( 1 \leq i \leq p \). Since \( G_X \) is the graph obtained from \( G \) by subdividing \( e_i \) \((1 \leq i \leq p)\), \( V(G_X) = V(G) \cup \{v(e_i) : 1 \leq i \leq p\} \), and \( E(G_X) = (E(G) - X) \cup \{u_iv(e_i), v(e_i)v_j : 1 \leq i \leq p\} \).

Case 1. \( |X \cap E(T_1)| + |X \cap E(T_2)| = 0 \). Then \( T_1 + \bigcup_{i=1}^{p} \{u_iv(e_i)\} \) and \( T_2 + \bigcup_{i=1}^{p} \{v(e_i)v_j\} \) are two edge-disjoint spanning trees in \( G_X \) and so \( F(G_X) = 0 \leq 2 \).

Case 2. \( |X \cap E(T_1)| + |X \cap E(T_2)| = 1 \). Let \( e_2 = u_2v_2 \) be the edge in \( X \cap E(T_2) \). Then \( T' = T_2 - e_2 + \{u_2v(e_2), v(e_2)v_2\} \bigcup_{i \neq 2} \{v(e_i)v_j\} \) is a spanning tree in \( G_X \). To obtain another spanning tree which covers \( v(e_2) \), we can add an edge \( e' = u_1v(e_2) \) to \( G_X \). Then \( T_1' = T_1 + \{e'\} \bigcup_{i \neq 2} \{u_iv(e_i)\} \) is a spanning tree in \( G_X + e' \). Therefore, \( T_1' \) and \( T_2' \) are two edge-disjoint spanning trees in \( G_X + e' \). This shows that \( F(G_X) = 1 \leq 2 \).

Case 3. \( |X \cap E(T_1)| + |X \cap E(T_2)| = 2 \).

By (2) and (3), either \( |X \cap E(T_1)| = |X \cap E(T_2)| = 1 \), or \( |X \cap E(T_1)| = 0 \) and \( |X \cap E(T_2)| = 2 \). We prove \( F(G_X) \leq 2 \) for the case \( |X \cap E(T_1)| = |X \cap E(T_2)| = 1 \) here. The case \( |X \cap E(T_1)| = 0 \) and \( |X \cap E(T_2)| = 2 \) can be proved similarly.

Let \( e_1 \in X \cap E(T_1) \) and \( e_2 \in X \cap E(T_2) \). Then \( T'_1 = T_1 - e_1 + \{u_1v(e_1), v(e_1)v_1\} \bigcup_{i \neq 1} \{u_1v(e_i)\} \) is a tree containing \( V(G_X) - v(e_1) \), and \( T'_2 = T_2 - e_2 + \{u_2v(e_2), v(e_2)v_2\} \bigcup_{i \neq 2} \{v(e_i)v_j\} \) is a tree containing \( V(G_X) - v(e_1) \). Therefore, adding two new edges \( e' = u_1v(e_2) \) and \( e'' = v(e_1)v_2 \) to \( G_X \), we have two edge-disjoint spanning trees \( T_1' + e' \) and \( T_2' + e'' \) in \( G_X + \{e', e''\} \). This shows that \( F(G_X) \leq 2 \). The proof is complete. \( \Box \)

**Lemma 2.7.** Let \( G \) be a graph with \( \kappa'(G) \geq 3 \), and let \( X \subseteq E(G) \). Let \( G_X \) be the graph obtained from \( G \) by subdividing each edge in \( X \). If the reduction of \( G_X \) is \( K_{2,t} \), then each of the following holds:

(i) Every degree 2 vertex in \( G_X' \) is a vertex obtained by subdividing an edge in \( X \).

(ii) \( |X| \geq t \geq \kappa'(G) \), and \( X \) is an edge cut of \( G \).

(iii) There is a subset \( X_1 \subseteq X \) with \( t = |X_1| \) such that each path between the two vertices of degree \( t \) in \( K_{2,t} \) is obtained by subdividing an edge in \( X_1 \). Furthermore, \( G_X - X_1 \) has only two possible components (say \( H_1 \) and \( H_2 \)) such that \( V(G_X) = V(H_1) \cup V(H_2) \bigcup_{e \in E_1} \{v(e)\} \), and \( G_X' = K_{2,t} \) is obtained by contracting \( H_1 \) and \( H_2 \) (i.e., \( G_X' = (G_X/H_1)/H_2 = K_{2,t} \)).

**Proof.** Let \( E(G_X') = E(K_{2,t}) = \{uw_i, w_iv\} \) \((1 \leq i \leq t)\) where each \( w_i \) is a degree 2 vertex in \( G_X' \). Note that \( w_i \) is a trivial contraction, and (i) holds. Otherwise the two edges incident with \( w_i \) will form an edge-cut of \( G \), contrary to that \( \kappa'(G) \geq 3 \). Hence, each path \( uw_iw \) is obtained by subdividing an edge in \( X \) and so \( u \in X \).

Let \( E' = \{uw_i : 1 \leq i \leq t\} \). Then \( E' \) is an edge-cut of \( G_X' \). Since each path \( uw_iw \) in \( G_X \) is obtained by subdividing an edge \( e \in X \subseteq E(G) \), we have an edge set \( X_1 \subseteq X \) such that each edge in \( X_1 \) corresponding to a path \( uw_iw \) in \( G_X \), and \( |X_1| = |E'| = t \). Therefore, \( X_1 \) is an edge cut in \( G \). Since \( X_1 \subseteq X \), \( X \) is an edge-cut of \( G \) and \( |X| = |E'| = t \geq \kappa'(G) \).

Note \( V(G_X') = \{u, v, w_i : 1 \leq i \leq t\} \) where \( d(u) = d(v) = t \). Let \( H_1 \) be the preimage of \( u \), and let \( H_2 \) be the preimage of \( v \). Therefore, \( G_X' \) is obtained by subdividing each edge in \( X_1 \), and then contracting \( H_1 \) and \( H_2 \), respectively. Statement (iii) is proved. \( \Box \)

**Lemma 2.8.** Let \( G \) be an \( r \)-edge-connected graph \((r \geq 4)\). Let \( X \subseteq E(G) \). Let \( G_X \) be the graph obtained from \( G \) by subdividing each edge in \( X \). Let \( G_X' \) be the reduction of \( G_X \) and let \( V_r \) be the set of vertices of degree less than \( r \) in \( G_X' \). Let \( D_i = \{v \in V(G_X') : d(v) = i\} \) \((i \geq 2)\). If \( F(G_X') \geq 3 \), then each of the following holds:

(i) each vertex in \( V_r \) has degree 2 \((i.e., V_r = D_2)\) and \( |V_r| \leq |X| \).

(ii) \((r - 4)|V(G_X')| + 10 \leq (r - 2)|V_r| \leq (r - 2)|X| \).

(iii) \( 10 + (r - 4)|D_r| + (r - 3)|D_{r+1}| + \cdots + \leq 2|V_r| \leq 2|X| \).

**Proof.** Since the degree of each vertex \( u \) in \( V_r \) is less than \( r \), \( u \) must be a trivial contraction in \( G_X' \). Otherwise, the edges incident with \( u \) will form an edge cut with size less than \( r \), contrary to \( \kappa'(G) \geq r \). Therefore, \( V_r \subseteq V(G_X) - V(G) \),
a subset of the vertices obtained in the process of subdividing each edge in \( X \). Thus each vertex in \( V_r \) has degree 2 and
\[
|V_r| \leq |X|.
\]
(4)

Let \( c = |V(G'_X)| \). Since \( F(G'_X) \geq 3 \), by (iii) of Theorem 2.1,
\[
|E(G'_X)| = 2|V(G'_X)| - 2 - F(G'_X) \leq 2c - 5.
\]
Hence,
\[
\sum_{v \in V(G'_X)} d(v) = 2|E(G'_X)| \leq 4c - 10.
\]
(5)

Since \( \kappa'(G_X) \geq 2 \), \( \delta(G'_X) \geq 2 \). Then by (5)
\[
2|V_r| + r(c - |V_r|) \leq 2|V_r| + \sum_{v \notin V_r} d(v) = \sum_{v \in V(G'_X)} d(v) = 2|E(G'_X)| \leq 4c - 10.
\]
(6)

By (4), (6), and \( c = |V(G'_X)| \),
\[
(r - 4)|V(G'_X)| + 10 \leq (r - 2)|V_r| \leq (r - 2)|X|.
\]
(7)

By (6), and \( V(G'_X) = V_r \cup \bigcup_{i=r} D_i \),
\[
2|V_r| + r|D_r| + (r + 1)|D_{r+1}| + \cdots \leq 4(|V_r| + |D_r| + |D_{r+1}| + \cdots) - 10.
\]
Hence,
\[
10 + (r - 4)|D_r| + (r - 3)|D_{r+1}| + \cdots \leq 2|V_r| \leq 2|X|.
\]
\[\square\]

Lemma 2.9. Let \( G \) be a graph and let \( e_1, e_2 \in E(G) \) and let \( X \subseteq E(G) \). Let \( X_0 = X \cup \{e_1, e_2\} \). Let \( G_{X_0} \) be the graph obtained from \( G \) by subdividing each edge in \( X_0 \). Let \( v(e_1) \) and \( v(e_2) \) be the two vertices subdividing \( e_1 \) and \( e_2 \), respectively. Then

(i) If \( G_{X_0} \) has a spanning \((v(e_1), v(e_2))\)-trail, then \( G \) has a spanning \((e_1, e_2)\)-trail containing \( X \).

(ii) If \( G_{X_0} \) is collapsible, then \( G \) has a spanning \((e_1, e_2)\)-trail containing \( X \).

Proof. Follows from the definitions of collapsibility and \( G_{X_0} \). \[\square\]

3. The \( r \)-Eulerian-connected graphs

The Petersen graph and many other 3-edge-connected graphs have no spanning closed trails. Thus, for any \( r \geq 0 \), \( \psi(r) \geq 0 \). By Theorem 2.2, we know that \( \psi(0) = \theta(0) = 4 \). The following example shows that for \( r \geq 3 \), \( \psi(r) \geq \theta(r) \geq r + 1 \).

Example 1. Let \( r \geq 3 \) be an integer, and let \( n \) and \( m \) be two integers such that \( n \geq r + 1 \) and \( m \geq r + 1 \). Let \( G_1 = K_n \) with \( V(G_1) = \{u_1, u_2, \ldots, u_n\} \), and let \( G_2 = K_m \) with \( V(G_2) = \{v_1, v_2, \ldots, v_m\} \). Define the graph \( G \) to be the graph obtained from \( G_1 \) and \( G_2 \) by connecting \( G_1 \) and \( G_2 \) with the new edge set \( X = \{e_1, e_2, \ldots, e_r\} \) where \( e_i = u_i v_i \) for all \( i = 1, 2, \ldots, r \). Then \( G \) is an \( r \)-edge-connected graph. If \( r \) is even, then we choose \( u \) from \( G_1 \), and \( v \) from \( G_2 \). If \( r \) is an odd integer, then we choose \( u \) and \( v \) both from \( G_1 \). Then \( G \) has no spanning \((u, v)\)-trails containing all the edges of \( X \). This example also shows that \( G \) has no spanning \((e', e'')\)-trails containing all the edges of \( X \) for some pair of \( e', e'' \in E(G) \). See Fig. 1 below for the case \( r = 4 \) where \( X = \{e_1, e_2, e_3, e_4\} \) and \( G_1 \cong G_2 \cong K_5 \). This shows that \( \psi(r) \geq \theta(r) \geq r + 1 \). In the following, we will show that \( \psi(r) = \theta(r) = r + 1 \).

This example suggests the following necessary condition for \( r \) Eulerian-connected graphs, and the lower bounds for \( \psi(r) \), \( \theta(r) \) and \( \xi(r) \).
Theorem 3.0. Let $r \geq 3$. Then $\psi(r) \geq \theta(r) \geq r + 1$ and $\zeta(r) \geq r + 1$. Furthermore, if $G$ is an $r$-Eulerian-connected graph, then $G$ is $(r + 1)$-edge-connected.

Proof. By way of contradiction, suppose that the edge-connectivity of $G$ is $k \leq r$. Let $X$ be an edge cut with $|X| = k$ and let $H_1$ and $H_2$ be two components of $G - X$. If $|X| = k$ is even, we can choose a vertex $u$ from $H_1$ and a vertex $v$ from $H_2$. Then $G$ has no spanning $(u, v)$ trail that contains $X$, a contradiction. If $|X| = k$ is odd, then we can choose a vertex $u$ from $H_1$. Since $X$ odd number of edges, $G$ does not have a closed trail that starts and ends at $u$ containing $X$, a contradiction again. □

For a real number $x$, let $\lfloor x \rfloor$ be the largest integer that is less than or equal to $x$.

Theorem 3.1. Let $r \geq 4$ be an integer and let $k = \lfloor \frac{r}{2} \rfloor$. Let $G$ be an $r$-edge-connected graph and let $X \subseteq E(G)$ with $|X| \leq r + k - 2$. It then one of the following holds:

(i) $G_X$ is collapsible, or
(ii) $X$ is an edge cut of $G$ and $|X| \geq r$.

Proof. Let $X \subseteq E(G)$ with $|X| \leq r + k - 2$. Define $G_X$ as before and assume that $G_X$ is not collapsible. We will show that the reduction $G'_X$ is $K_{2,t}$ with $t \geq 2$ first. Consider the following two cases:

Case 1. $r$ is even. Then $r = 2k$, and $|X| \leq 3k - 2$.

Since $|X| \leq 3k - 2$, we can choose a subset $X_1$ of $X$ and let $X_2 = X - X_1$, such that $|X_1| \leq k$ and $|X_2| \leq 2k - 2$. By Theorem 2.3, $G - X_1$ has $k$-edge-disjointed spanning trees. Then by Lemma 2.6(iv), $F((G - X_1)_{X_2}) \leq 2$. By Lemma 2.6(iii), $F(G_X) \leq F((G - X_1)_{X_2}) \leq 2$. Since $G_X$ is not collapsible, by Theorem 2.5, $G_X \in \{K_2, K_{2,t}\} (t \geq 1)$. Since $G$ is $r$-edge-connected ($r \geq 4$), $G_X$ is 2-edge-connected. Therefore, $G'_X = K_{2,t}$ ($t \geq 2$).

Case 2. $r$ is odd. Then $r = 2k + 1$ and $|X| \leq 3k - 1$.

Let $X_1$ be a subset of $X$ and let $X_2 = X - X_1$ such that $|X_1| \leq k + 1$ and $|X_2| \leq 2k - 2$. By Corollary 2.4, $G - X_1$ has $k$-edge-disjointed spanning trees. By Lemma 2.6(iii) and (iv), $F(G_X) \leq F((G - X_1)_{X_2}) \leq 2$. Using the same argument for the case 1 above, we have $G'_X = K_{2,t} (t \geq 2)$.

Therefore, by Lemma 2.7, Theorem 3.1 is proved. □

From the proof of Theorem 3.1, we have the following:

Theorem 3.1'. Let $r \geq 4$ be an integer and let $k = \lfloor \frac{r}{2} \rfloor$. Let $G$ be an $r$-edge-connected graph. Let $X \subseteq E(G)$ with $|X| \leq r + k - 2$ and let $G_X$ be the graph obtained from $G$ by subdividing every edge in $X$. Let $G'_X$ be the reduction of $G_X$. Then exactly one of the following holds

(i) $G_X$ is collapsible, or
(ii) $G_X$ can be contracted to $K_{2,t}$ (i.e. $G'_X = K_{2,t}$) in such a way that each degree vertex in $K_{2,t}$ is a trivial contraction and $r \leq t \leq |X|$.

Theorem 3.2. Let $r \geq 4$ be an integer and let $k = \lfloor \frac{r}{2} \rfloor$. Let $G$ be an $r$-edge-connected graph. Let $X \subseteq E(G)$ with $|X| \leq r + k - 2$. Then one of the following holds

(i) for any even subset $R \subseteq V(G)$, $G$ has a spanning $R$-trail $H_R$ such that $X \subseteq E(H_R)$, or
(ii) $X$ is an edge cut of $G$ and $|X| \geq r$.  

Fig. 1.
Theorem 3.5. Let $|X| \leq r + k - 2$. If $X$ is not an edge cut of $G$, then $G$ has a spanning $(R, X)$-trail for any even subset $R \subseteq V(G)$. Theorem 3.2 follows from Theorem 3.1. \hfill $\Box$

Corollary 3.3. Let $r \geq 4$ be an integer, and let $k = \left\lceil \frac{r}{2} \right\rceil$. Let $G$ be an $r$-edge-connected graph. Let $X \subseteq E(G)$ with $|X| \leq r + k - 2$. If $X$ is not an edge cut of $G$, then $G$ has a spanning $(R, X)$-trail for any even subset $R \subseteq V(G)$. 

Proof. Following Theorem 3.1 and Lemma 2.6 immediately. \hfill $\Box$

Corollary 3.4. Let $r \geq 3$. Then $G$ is strongly $r$-Eulerian-connected if and only if $G$ is $(r + 1)$-edge-connected.

Proof. The necessary condition follows from Theorem 3.0. For the sufficient condition, let $X \subseteq E(G)$ with $|X| \leq r$. Then $|X| < \kappa'(G) = r + 1$. $X$ is not an edge cut of $G$ and by Theorem 3.2, the statement holds. \hfill $\Box$

Theorem 3.5. Let $r \geq 0$. Then

$$
\psi(r) = \begin{cases} 4 & \text{if } 0 \leq r \leq 2, \\ r + 1 & \text{if } r \geq 3. 
\end{cases}
$$

Proof. Since there exist 3-edge-connected graphs which are not supereulerian, $\psi(r) \geq \theta(r) \geq 4$ for $r \geq 0$. By Theorem 3.1, if $G$ is 4-edge-connected, then any edge set $X$ with $|X| \leq 2$ cannot be an edge cut of $G$. Therefore $G_X$ is collapsible, and so $\theta(r) = \psi(r) \leq 4$ if $r \leq 2$. For $r \geq 3$, it follows from Corollary 3.4 that $\psi(r) = \theta(r) = r + 1$. \hfill $\Box$

Corollary 3.6 (Lai [12]). Let $r \geq 0$ be an integer. Then

$$
f(r) = \begin{cases} 4, & 0 \leq r \leq 2, \\ r + 1, & r \geq 3 \text{ and } r \text{ is odd}, \\ r, & r \geq 4 \text{ and } r \text{ is even}. 
\end{cases}
$$

Proof. Since there exist 3-edge-connected graphs that are not supereulerian, $f(r) \geq 4$. Since $f(r) \leq \theta(r)$, by Theorem 3.1, $f(r) = 4$ if $r \leq 2$. For $r \geq 3$, if $r$ is odd, Example 1 with an odd number $r$ shows that $f(r) = r + 1$. By Theorem 3.1, since $f(r) \leq \theta(r) \leq r + 1$, $f(r) = r + 1$ if $r$ is odd. If $r$ is even, by Theorem 3.1', for any $r$-edge-connected graph $G$ and any $X \subseteq E(G)$ with $|X| \leq r$, either $G_X$ is collapsible or the reduction $G_X' \cong K_{2,r}$. Since $K_{2,r}$ is supereulerian when $r$ is even and all collapsible graphs are supereulerian, $G_X$ is supereulerian. Then by Lemma 2.6(ii), $G$ has a spanning Eulerian subgraph $H$ with $X \subseteq E(H)$. Therefore, $f(r) = r$ if $r$ is even. \hfill $\Box$

Corollary 3.6 implies that if $G$ is 4-edge-connected, then for any $X \subseteq E(G)$ with $|X| \leq 4$, $G$ has a spanning Eulerian subgraph $H$ such that $X \subseteq E(H)$. Here we have:

Theorem 3.7. Let $G$ be 4-edge-connected graph. Let $X \subseteq E(G)$ with $|X| \leq 5$. Let $G_X$ be the graph obtained from $G$ by subdividing each edge in $X$. Let $D_i = \{v \in V(G_X') \mid d(v) = i\}$ ($i \geq 2$). Then one of the following holds

(i) $G_X$ is collapsible, or
(ii) $X$ contains an edge cut $X_1$ with $|X_1| = t \geq 4$ such that $G - X_1$ has only two components ($H_1$ and $H_2$), which are collapsible. Furthermore, $G_X$ is contractible to $K_{2,t}$ by contracting $H_1$ and $H_2$ into the two degree $t$ vertices in $K_{2,t}$, or
(iii) $G_X'$ is an Eulerian graph with $V(G_X') = D_2 \cup D_4$ and $|D_2| = 5$.

Proof. Let $G_X'$ be the reduction of $G_X$. If $G_X' = K_1$, then $G_X$ is collapsible and we are done for this case. In the following we will assume that $G_X'$ is not trivial. Since $G$ is 4-edge-connected, $G_X$ is 2-edge-connected. Since $\kappa(G_X') \geq \kappa(G_X)$, $G_X'$ is 2-edge-connected.

Case 1. $|F(G_X')| \leq 2$.

By Theorem 2.5, and $\kappa'(G_X) \geq 2$, $G_X' \cong K_{2,t}$, for some $t \geq 2$. By Lemma 2.7, $|X| \geq t \geq 4$. Hence, (ii) of Theorem 3.7 holds.
This implies that $|D_i| = 0$ for all $i \geq 5$ and $|D_2| = 5$. Therefore, each vertex in $V(G'_X)$ has degree 2 or 4. Hence, $G'_X$ is Eulerian and $|D_2| = 5$.

**Corollary 3.8.** Let $G$ be a 4-edge-connected graph. Let $X \subseteq E(G)$ with $|X| \leq 5$. Let $G_X$ be the graph obtained from $G$ by subdividing each edge in $X$. Then either $G$ has a spanning Eulerian subgraph $H$ such that $X \subseteq E(H)$, or $G_X$ is contractible to $K_{2,5}$ in such a way that each path between the two vertices of degree 5 is obtained by subdividing an edge in $X$.

**Proof.** This follows from Theorem 3.7 and Lemma 2.9. □

### 4. The $r$-edge-Eulerian-connected graphs

We will need the following lemma.

**Lemma 4.0.** Let $G$ be a 3-edge-connected graph. Let $X \subseteq E(G)$ and let $e'$, $e'' \in E(G)$. Let $X_0 = X \cup \{e', e''\}$ and let $G_{X_0}$ be the graph obtained from $G$ by subdividing each edge in $X_0$. Suppose that $G'_{X_0} = K_{2,t}$ where $t \geq 3$. If $t > |X|$, then $G$ has a spanning $(e', e'')$-trail $H$ such that $X \subseteq E(H)$.

**Proof.** Let $u$ and $v$ be the two vertices in $K_{2,t}$ with $d(u) = d(v) = t$. By Lemma 2.7, there is an edge set $X_1 \subseteq X_0$ such that each length 2 path between $u$ and $v$ in $K_{2,t}$ is obtained by subdividing an edge in $X_1$. Then $|X_1| = t$. Let $E_1 = E(G'_{X_0}) = E(K_{2,t})$. By Lemma 2.7, $G_{X_0} - E_1$ has two collapsible subgraphs $(H_1$ and $H_2$) such that $V(G_{X_0}) = V(H_1) \cup V(H_2) \cup e \in X_1 \{v(e)\}$. Let $e' = x_0'y_0'$, $e'' = x''_0'y''_0$ and let $x_0', x''_0', y_0', y''_0 \in V(H_1)$ and $y_0', y''_0 \in V(H_2)$. Since $t > |X|$, at least one of the edges in $\{e', e''\}$ is included in $X_1$. For each $e \in \{e', e''\}$, $P_e$ is defined as a path obtained by subdividing edge $e$.

For each $H_i$, $(i = 1, 2)$, define

$$U_0(H_i) = \{v \in V(H_i) : v \text{ is incident with odd number of edges in } E_1 \setminus \{P_{e'}, P_{e''}\}\}.$$

Note that $|U_0(H_1)|$ is odd if and only if $|U_0(H_2)|$ is odd. Since $H_i$ is collapsible, for any even subset $R_i \subseteq V(H_i)$, there is a spanning connected subgraph $\Gamma_i$ with $O(\Gamma_i) = R_i$ $(i = 1, 2)$. In the following we will show that a spanning $(v(e'), v(e''))$-trail $\Gamma$ can be constructed from $\Gamma_1$ and $\Gamma_2$ by adding all the edges in $E_1$ and an edge $e_{\Gamma_1}$ to connect $v(e')$ (or an edge $e_{\Gamma_2}$ to connect $v(e'')$, or both) such that $O(\Gamma) = \{v(e'), v(e'')\}$.

**Case 1.** Both $e'$ and $e''$ are in $X_1$.

Note that $G$ may not be simple and we may have three possible situations:

(a) $x'_0 = x''_0$ and $y'_0 = y''_0$,

(b) $x'_0 = x''_0$ and $y'_0 \neq y''_0$,

(c) $x'_0 \neq x''_0$ and $y'_0 = y''_0$.

The following Tables 1–3 show the selections of the even subset $R_i \subseteq V(H_i)$ for $\Gamma_i$ and $e_{\Gamma_i}$ $(i = 1, 2)$ for all possible cases.

For each case with the selection of $R_1$, $R_2$, $e_{\Gamma_1}$ and $e_{\Gamma_2}$, define

$$\Gamma = G_{X_0}[E(\Gamma_1) \cup E(\Gamma_2) \cup E_1 \cup \{e_{\Gamma_1}, e_{\Gamma_2}\}]$$

By the definition of $\Gamma$, $V(\Gamma) = V(\Gamma_1) \cup V(\Gamma_2) \cup \{v(e') \cup v(e'')\}$, and $v(e')$ and $v(e'')$ have degree 1 in $\Gamma$. Since $\Gamma_i$ is a connected spanning subgraph of $H_i$, $V(\Gamma_i) = V(H_i)$ $(i = 1, 2)$. $\Gamma_1$ and $\Gamma_2$ are connected by the paths in $E_1$, and $v(e')$ and $v(e'')$ are connected to $\Gamma_i$ by $e_{\Gamma_i}$. Thus, $V(\Gamma) = V(G_{X_0})$ and $\Gamma$ is a connected spanning subgraph...
of $G_{X_0}$. To show that $O(I) = \{v(e'), v(e'')\}$, we can check each case listed in Tables 1–3. For instance, with the cases in Table 1, if $v \notin R_1 \cup R_2$, $v$ has even degree in $I_1$ or $I_2$ or $v$ has degree 2 as a vertex obtained by subdividing an edge in $X_1$. If $v \in R_1$ and $v \neq x_0$ (or $v \in R_2$ and $v \neq y_0$), then since odd number of edges adjacent with $v$ in $E_1$ are added, $v$ has an even degree in $I$. If $v = x_0$ (or $y_0$), by the definition of $e_{I_1}$ and $e_{I_2}$, $x_0$ has an even degree in $I$. Hence, $O(I) = \{v(e'), v(e'')\}$, and $\Gamma$ is a spanning $(v(e'), v(e''))$-trail in $G_{X_0}$. By Lemma 2.9, $G$ has a spanning $(e', e'')$-trail containing $X$.

**Case 2.** One of $e'$ and $e''$ is in $X_1$ (say $e' \in X_1$).

Since $e'' \notin X_1$, we may assume that the path obtained by subdividing $e''$ is in $H_1$. Then $v(e'') \in V(H_1)$. For this case, we only need to choose $e_{I_1}$ to connect $v(e')$ in $\Gamma$.

For each case in Table 4, define

$$\Gamma = G_{X_0}[E(I_1) \cup E(I_2) \cup E_1 \cup \{e_{I_1}\}].$$

Therefore, $\Gamma$ is a spanning connected subgraph of $G_{X_0}$ such that $O(I) = \{v(e'), v(e'')\}$. The Lemma is proved.

In [14], Zhan proved the following:

**Theorem 4.1 (Zhan [14]).** If $G$ is a 4-edge-connected graph, then for any edges $e_1, e_2 \in E(G)$ there is a spanning $(e_1, e_2)$-trail in $G$. 

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Theorem 4.1 can be improved.

**Theorem 4.2.** Let \( r \in \{3, 4\} \). If \( G \) is an \((r + 1)\)-edge-connected graph, then for any \( X \subseteq E(G) \) with \(|X| \leq r - 1\), and for any \( e_1, e_2 \in E(G) \), \( G \) has a spanning \((e_1, e_2)\)-trail \( H \) in \( G \) such that \( X \subseteq E(H) \).

**Proof.** Let \( X_0 = X \cup \{e_1, e_2\} \). Let \( G_{X_0} \) be the graph obtained from \( G \) by subdividing each edge in \( X_0 \). Since \( r \in \{3, 4\} \), \( k = \lfloor (r + 1)/2 \rfloor = 2 \). Then \(|X_0| \leq |X| + 2 \leq r + 1 = (r + 1) + k - 2 \). By Theorem 3.1', either \( G_{X_0} \) is collapsible or \( G_{X_0} \) is contractible to \( K_{2, t} \) with \( t \geq r \). If \( G_{X_0} \) is collapsible, then by Lemma 2.9, \( G \) has a spanning \((e_1, e_2)\)-trail containing \( X \). If \( G_{X_0} \) is contractible to \( K_{2, t} \) with \( t \geq 4 \), since \( t \geq r \geq |X| \), by Lemma 4.0, \( G \) has a spanning \((e_1, e_2)\)-trail containing the edge set \( X \). \( \square \)

For graphs with edge-connectivity at least 5, we have

**Theorem 4.3.** Let \( G \) be an \((r + 1)\)-edge-connected graph \((r \geq 4)\). Let \( X \subseteq E(G) \) with \(|X| \leq r \). Then \( G \) is an \( r \)-edge-Eulerian-connected.

**Proof.** Let \( e_1 \) and \( e_2 \) be two arbitrary edges in \( G \) and let \( X_0 = X \cup \{e_1, e_2\} \). Let \( G_{X_0} \) be the graph obtained from \( G \) by subdividing each edge in \( X_0 \).

Case 1. \( r \geq 5 \).

Then \( r + 1 \geq 6 \), and so \( k = \lfloor (r + 1)/2 \rfloor \geq 3 \). Then \(|X_0| \leq |X| + 2 \leq r + 2 \leq (r + 1) + k - 2 \). By Theorem 3.1', either \( G_{X_0} \) is collapsible or \( G_{X_0} \) is contractible to \( K_{2, t} \) with \(|X_0| \geq t \geq (r + 1) \). By Lemma 2.9 and Lemma 4.0, both cases imply that \( G \) has a spanning \((e_1, e_2)\)-trail \( H \) such that \( X \subseteq E(H) \). Theorem 4.3 is proved for this case.

Case 2. \( r = 4 \).

Then \( G \) is 5-edge-connected and \(|X_0| \leq 6 \). Let \( G'_{X_0} \) be the reduction of \( G_{X_0} \). If \( F(G'_{X_0}) \leq 2 \), then \( G_{X_0} \) is either collapsible or contractible to \( K_{2, t} \) with \( t \geq (r + 1) \) and so we are done. Next we assume that \( F(G'_{X_0}) \geq 3 \).

**Claim.** If \( v \in D_2 \subseteq V(G'_{X_0}) \), then the degree of each of the two neighbors of \( v \) is greater than 2.

Since \( \delta(G) \geq \kappa'(G) \geq 5 \), each vertex of degree 2 in \( G'_{X_0} \) is obtained by subdividing an edge in \( X_0 \). If a degree vertex has a neighbor which is also degree 2, then this will contradict the definition of \( G_{X_0} \).

By Lemma 2.8, we have

\[
|V(G'_{X_0})| + 10 \leq 3|D_2| \leq 3|X_0|.
\]  
(8)

If \(|D_2| \leq 5\), then by (8), \(|V(G'_{X_0})| \leq |D_2| \leq 5\), contrary to the claim above. Therefore, \(|D_2| = |X_0| = 6\). By (8) and \(|D_2| = 6\),

\[
|V(G'_{X_0})| \leq 8.
\]

Therefore, \( G'_{X_0} \) is a 2-edge-connected graph with 6 vertices of degree 2 and at most two vertices of degree at least 5. By the claim above, vertices of degree 2 are not adjacent to each other. Therefore, \( G'_{X_0} = K_{2,6} \), contrary to \( F(G'_{X_0}) \geq 3 \). The theorem is proved. \( \square \)

Let \( r \) be an integer. Theorem 4.2 shows that if \( G \) is 4-edge-connected, then \( G \) is 2-edge-Eulerian-connected. If \( r \geq 4 \) and if \( G \) is \((r + 1)\)-edge-connected, then \( G \) is \( r \)-edge-Eulerian-connected. Combining Theorems 4.2, 4.3 and 3.0, we have:
Corollary 4.4. Let \( r \geq 0 \) be an integer. Then
\[
\zeta(r) = \begin{cases} 
4, & 0 \leq r \leq 2, \\
r + 1, & r \geq 4.
\end{cases}
\]

Remark. The case \( \zeta(3) \) is still open. Theorem 4.2 implies that if \( G \) is 5-edge-connected, then \( G \) is 3-edge-Eulerian-connected, and so \( \zeta(3) \leq 5 \). We conjecture that \( \zeta(3) = 4 \). The following theorem provides some supports for this conjecture.

Theorem 4.5. Let \( G \) be a 4-edge-connected graph and let \( X \subseteq E(G) \) with \( |X| \leq 3 \). For any two adjacent edges \( e' \) and \( e'' \), \( G \) has a spanning \((e', e'')\)-trail \( H \) such that \( X \subseteq E(H) \).

Proof. Let \( X_0 = X \cup \{e', e''\} \). Let \( G_{X_0} \) be the graph obtained from \( G \) by subdividing each edge in \( X_0 \). Let \( v(e') \) and \( v(e'') \) be the two vertices obtained in the process of subdividing \( e' \) and \( e'' \). If \( G_{X_0} \) is collapsible, then \( G_{X_0} \) has a spanning connected subgraph \( H \) such that \( O(H) \) is \( v(e'), v(e'') \). By Lemma 2.9, \( G \) has a spanning \((e', e'')\)-trail containing \( X \). We are done in this case. Next, we assume that \( G_{X_0} \) is not collapsible.

Let \( G'_{X_0} \) be the reduction of \( G_{X_0} \). By Theorem 3.7, either \( G'_{X_0} = K_{2,1} \), with \( t \geq 4 \) or \( G'_{X_0} \) is Eulerian with \( V(G'_{X_0}) = D_t \cup D_4 \), where \( D_t \) is the set of vertices of degree \( t \) in \( G'_{X_0} \). If \( G'_{X_0} = K_{2,1} \) with \( t \geq 4 \), then by Lemma 4.0, \( G \) has a spanning \((e', e'')\)-trail such that \( X \subseteq E(H) \). We are done for this case.

For the case that \( G'_{X_0} \) is Eulerian, let \( v \) be the vertex incident with both \( e' \) and \( e'' \). Suppose \( v(e') \) and \( v(e'') \) are the two vertices obtained in the process of subdividing \( e' \) and \( e'' \). Then \( G'_{X_0} - \{e', e''\} \) is connected. Otherwise, \( \{e', e''\} \) is an edge cut of \( G \), contrary to that \( G \) is 4-edge-connected. Therefore, \( G'_{X_0} - \{e', e''\} \) is a connected graph with only two odd degree vertices at \( v(e') \) and \( v(e'') \). Let \( U_4 = \{u \in D_4 \mid u \) is a non-trivial contraction.\} For each vertex \( u \in U_4 \), let \( H(u) \) be the preimage of \( u \) in \( G'_{X_0} \). Then \( H(u) \) is a collapsible graph. Let \( V_4 = \{x \in E(V(H(u))) : x \) is incident with odd number of edges in \( G'_{X_0} - \{e', e''\}\}.\)

Since \( d(u) \) in \( G'_{X_0} - \{e', e''\} \) is even, \( |V_4| \) is even or 0. Since \( H(u) \) is collapsible, \( H(u) \) has a spanning connected subgraph \( F_u \) such that \( O(F_u) = V_4 \). Let \( E_0 = E(G_{X_0}) - \{e', e''\} \) and let \( \Gamma = G_{X_0} \bigcup_{u \in U_4} E(F_u) \cup E_0 \).

Then \( \Gamma \) is a spanning connected subgraph of \( G_{X_0} \) such that \( O(\Gamma) = v(e'), v(e'') \). Therefore, \( G_{X_0} \) has a spanning \((v(e'), v(e''))\)-trail. By Lemma 2.9, \( G \) has a spanning \((e', e'')\)-trail containing \( X \). The proof is complete. \( \square \)

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References