SCRAMBLE AND RETROGRADE ANALYSIS

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Reading Mike Keith’s article “Scramble-Tile Double Word Squares and Rectangles” (WW Feb 2011) led me to ask the following question: which of the word grids he gave are actually legal positions in a game of Scramble? Chess problemists use the term “retrograde analysis” for the study of how chess positions might have arisen in a legal game; this concept is of course equally applicable to other games, including Scramble.

I should say, for the benefit of readers who may wish to pursue any of the questions I ask, that when writing this article I have not attempted to consult any online word lists; all the words used here are either words I know or words used in Mike Keith’s article.

The rules of Scramble are easy to axiomatise. (The following includes references to fixed numbers – 7, 15, 100 – which are correct for the standard game of Scramble, but which can in principle be changed.)

1. The alphabet is a set of symbols called letters. A word is a sequence of at least 2 letters. (Note that 1-letter words play no role in Scramble.) A subword of a word is a sequence of at least 2 letters obtained by removing (possibly zero) letters from the beginning and end; it is proper if at least one letter is removed. The lexicon is a set of words (those which are allowed in Scramble). The ensemble is the collection of letters, each with a given multiplicity, which can be used in a Scramble position. Apart from a letter for each lettered tile, the ensemble includes “wild cards” – the equivalent of the blank tiles which can represent any card – and its total size is 100.

2. A polyomino is a geometrical figure, made up of unit squares, where any two squares are connected by a sequence in such a way that adjacent squares in the sequence share a common (full length) edge. The board is a particular polyomino, a square of side 15, whose central square plays a distinguished role. A polyomino is possible if it has between 2 and 100 squares and can be placed within the board so as to include the central square. In a polyomino a chain of unit squares consists of an \( n \times 1 \) or \( 1 \times n \) rectangle (with \( n \geq 2 \)) entirely within the polyomino, and such that if extended (in either of the two possible directions) to an \( n + 1 \times 1 \) or \( 1 \times n + 1 \) rectangle then it is no longer entirely within the polyomino.

3. A crossword is a polyomino, with left-right or up-down directions defined on every chain in a compatible way, in which each unit square is labelled with a letter. It is possible if the polyomino is possible, and if every chain, read left to right or top to bottom, forms a word within the lexicon. A word is possible if its letters fall within the ensemble (i.e. excluding words like RAZZMATAZZ), and if it fits within the board (i.e. has at most 15 letters).

4. An initial move consists of the setting up of a chain, of length at most 7, which is labelled and oriented to form a word within the lexicon and positioned on the board in such a way as to include the central square. A move is either an initial move, or an operation that takes a crossword \( A \) to a crossword \( B \), both possible and with the same orientation, by inserting between 1 and 7 squares, labelled with letters, in such a way that there is a single word of \( B \) which contains all the inserted letters. An unmove is the reverse of a move.

5. A crossword is legal if it is the result of an initial move followed by a (possibly empty) sequence of moves. A polyomino is legal if it can be labelled with letters so as to form a legal crossword. A word is legal if there is a legal crossword in which it appears as a word. A polyomino is true if any crossword obtained by labelling it is legal if it is possible, and if this remains so with an arbitrary lexicon. (That means essentially that it can be built up without using any proper subwords of the words that appear in it.) A grid is a rectangular polyomino in which each dimension is at least 2. Without loss of generality we can assume that its length – the distance from left to right – is at least as great as its height – the distance from top to bottom. (This was the case with all the grids depicted in Mike Keith’s article.)

Note the following:

(a) I am assuming that all 100 tiles can be placed on the board. This is not true of a real game with at least 2 players, as a game ends as soon as any player has no tiles left. In general, in an \( n \)-player game, the maximum number of tiles that can be placed before the game ends is \( 101 - n \). But in a solitaire game all 100 tiles can be placed (as the above formula implies).
Q1: Is there any other way in which a crossword can be legal in a solitaire game but not in a game with at least 2 players? To show how this might happen, imagine a 2-player game in which 86 tiles had been placed. This would leave each player with 7 tiles. It would therefore be impossible to extend this game by three 4-letter moves. Can one find a crossword which cannot be reached by a different sequence of moves?

(b) The above does not involve itself with matters of scoring, or the possibility of “pass moves” in which a player exchanges letters. Neither is relevant to the question of which positions can be the result of a legal game.

To start the ball rolling (and illustrate the definition of trueness) we prove that any grid of size at most $8 \times 3$ is true. In fact, we can state a stronger result: a crossword based on such a grid can be positioned with any letter being at the central square of the board, and however this is done the crossword will be legal. The proof is simple: choose as the initial move the column (of length 2 or 3) containing the central square; then complete the top row; then complete the bottom row; then, if the height of the grid is 3, complete the middle row. It can easily be seen that the only words created are those that appear in the full grid.

This means that the smallest possible grid which might be possible but not legal is $4 \times 4$. It is easy to see that such a grid is not true, because any possible unmove must create at least one new word (in fact more than one).

Q2: Is there a $4 \times 4$ grid which is possible but not legal? If not, what is the smallest grid of this type?

Mike Keith’s article included the “heterogrammatic” $4 \times 4$ grid shown below (Table 1, centre). This grid is legal: one can start by removing the B, then the S, E and W, then the rest is easy.

There are polyominoses smaller than the $4 \times 4$ grid which are not true; this can happen when there are words too long to be playable in a single move, but it can also happen for the same reason as with a $4 \times 4$ grid. I am pretty sure that the smallest case of the latter is a $4 \times 4$ square with its corners removed.

Q3: Is there a crossword based on this polyomino which is possible but not legal? If this is to happen, none of the proper subwords of its 4-letter words may appear in the lexicon. If there is no crossword of this type, what is the smallest crossword consisting of more than one word that is possible but not legal?

Mike Keith said that there was a unique $7 \times 7$ grid which was possible (and where no two words were the same). This grid is reproduced below (Table 1, left). I was delighted to find that it is legal: it can be created by the following sequence of moves:

A: Start with the word SNIEZER.
B: In any order, lay the words SNIP, EIDOLIC, ZONATED and RESECTS in the first, third, fifth and bottom rows.
C: In any order, lay the words NOISOME, PHORATE, MI and RESCESD in the second, fourth, sixth and last columns.
D: Complete the words ESFIRE, EMINENT, POLITIC and INENES in that order.

However, I was unable to find a legal game leading to his “high scoring” $7 \times 5$ grid (Table 1, right)

Let me conclude with some further questions for readers:

Q4: Find a polyomino that is possible but not legal. Mike Keith’s article would have proved that this was true of a grid of size larger than those he constructed but for his requirement that all words within any of his grids be distinct. However I am pretty sure that such a polyomino exists.

Q5: Is every word legal if it is possible? Note that even a 15-letter word can be built up independently of whether appropriate subwords are in the lexicon; the relevant unmove is to remove the letters in the 7 even positions and leave the letters in the 8 odd positions. However it is not obvious, though it seems likely, that a crossword can always be chosen in such a way that this unmove is to a possible position and can be followed by further unmoves which prove the crossword legal.
A PUZZLE

across
1 Spring  
4 Winning _  
8 Won a tennis serve  
12 Adj: suff.  
13 King _  
14 Tinkers, Evers and Chantee  
15 Boxing  
18 Tie score  
19 Thus. Lat.  
20 Button or Fleming  
23 Croquet hoop
26 Sailor's aid  
27 Sheepish remark  
28 Adam's rib  
29 Bows  
32 __ jongg  
33 America's _  
34 Abounding  
35 Belonging to 1964, 1500 m
37 Puzzle theme
38 King: Fr.  
39 100 m race, e.g.  
40 Brit. Grand National entrant  
41 Back talk  
42 Abounding court zone  
43 Belonging to the towel
44 Gold Medalist
45 Island: Fr.

down
1 Small drink  
2 Golfer's goal  
3 15A name  
4 Football team  
5 Postpone  
6 Ten yard ___  
7 Energy unit  
8 Loft  
9 Brit. game player  
10 The same: Ger.  
11 __leg  
12 Gr. letters  
13 Chin. dynasty  
14 Bridge coups  
15 Islamic book  
16 Crossbow expert: arch.  
17 43D need  
18 Happening  
19 Hikers' gear  
20 Hit: sl. ___  
21 Team vehicle  
22 Hula Bowl welcome  
23 Football penalty  
24 Conclusion  
25 Soil deposit  
26 Bone: pref. ___-sea fishing  
27 The way: Ch. ___ Vegas  
28 Lodge type  
29 Moray  
30 Whiskey